ECE517, Fall 2023, homework 2

submit by September 28, 2023

1 Sine

4pts Find a regulator for the control system

$$\dot{y} = \sin ay + u, y, u \in \mathbb{R}; a \in \mathbb{R}, \text{unknown}.$$

Prove that it has the desired properties: $y \to 0$, bounded control.

2 Taylor in higher dimension

Versions of Taylor expansion working in higher dimension are useful in many situations.

(a) **2pts** For a C^1-smooth function $f : \mathbb{R}^n \to \mathbb{R}^n$, prove that there exists a continuous matrix-valued function $G : \mathbb{R}^n \times \mathbb{R}^n \to \mathtt{Mat}(n \times n)$, such that

$$f(y) = f(x) + G(x, y - x)(y - x)$$

for all $x, y \in \mathbb{R}^n$.

(b) **2pts** We say that a C^1 function $f : \mathbb{R}^n \to \mathbb{R}$ is M-*smooth* if its Jacobian matrix is M-Lipschitz:

$$\|J(x) - J(y)\| \leq M|x - y|, \qquad \forall x, y \in \mathbb{R}^n$$

(where

$$J := \frac{\partial f}{\partial x}$$

is the Jacobian of f, $\| \cdot \|$ denotes operator norm, and $| \cdot |$ denotes the usual Euclidean norm). Prove that if f is M-smooth, then

$$f(y) - f(x) - J(x)(y - x) \leq \frac{M}{2}|x - y|^2.$$

Hint: differentiate the C^1 function $F(t) := f(x + t(y - x))$, $t \in [0, 1]$.

3 Backstepping

5pts Assume that V_0 and k_0 are (strict) Lyapunov function and stabilizing feedback control (satisfying $k_0(0) = 0$) for the nonlinear system

$$\dot{x} = f(x, u)$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, where $f(\cdot, \cdot)$ satisfies $f(0, 0) = 0$ and is C^1 in both arguments.

Use the backstepping method to find a control Lyapunov function $V_1(x, \xi)$ and stabilizing feedback control $k_1(x, \xi)$ for the system

$$\begin{aligned} \dot{x} &= f(x, \xi) \\ \dot{\xi} &= h(x, \xi) + u \end{aligned}$$

with $x \in \mathbb{R}^n$ and $\xi, u \in \mathbb{R}^m$.
Hint: use 2(a).

4 Bounded Input → Output

5pt For the linear system

$$\dot{x}(t) = Ax(t) + Bu(t), x \in \mathbb{R}^n, u \in \mathbb{R}^m,$$

with Hurwitz A, prove that for some C,

$$\int_0^T |x(t)|^2 dt \le C \int_0^t |u(t)|^2 dt$$

Hint: Use the Lemma we proved in class: if a nonnegative kernel K is integrable, $\int_{-\infty}^{\infty} K(u)du < M$, then

$$\left| \int_0^T \int_0^T z(u) K(u - v) z(v) du dv \right| \le M \int_0^\infty |z(u)|^2 du.$$