# Coefficient asymptotics of algebraic multivariable generating functions [1]

**Abstract:**

Analytic combinatorics in several variables (ACSV) seeks to extract the asymptotic behavior of coefficients from a generating function in several variables. As much as possible, the extraction should be automatic, taking as input some finite specification of the generating function. A complexity hierarchy of specifiable generating functions usually begins with rational functions, then goes up to algebraic functions, D-finite functions, and perhaps beyond. Already the first two classes contain the majority of combinatorial problems for which any kind of generating function has been written down. For examples of rational multivariate generating functions in combinatorics see [PW08]. For examples of algebraic multivarate generating functions, see [GMRW22]. ACSV machinery developed for rational generating functions uses integrals of residue forms over intersection cycles to provide asymptotics for coefficients via the multivariate Cauchy integral formula. By embedding the coefficient array for an algebraic generating function as the generalized diagonal of the coefficient array of a rational generating function with one more variable, the authors of [GMRW22] are able to reduce coefficient asymptotics for a class of algebraic generating functions to a previously solved problem involving a rational function in one more variable.

In this paper, we take a different approach, namely to write the Cauchy integral for an arbitrarily specified algebraic function directly as an integral over the defining surface for the algebraic function. This leads to a similar computation, without invoking the technical apparatus of residue forms and intersection cycles. We give examples of how to apply this method to the functions in [GMRW22] and compare the difficulty of our computations and transparency of our formulas to those presented in the online appendix to [GMRW22].

YULIY BARYSHNIKOV, UNIVERSITY OF ILLINOIS, DEPARTMENT OF MATHEMATICS, 273 ALTGELD HALL 1409 W. GREEN STREET (MC-382), URBANA, IL 61801, ymb@illinois.edu, PARTIALLY SUPPORTED BY ARMY RESEARCH OFFICE (ARO) UNDER THE AWARD W911NF1810327.

KAITIAN JIN, UNIVERSITY OF PENNSYLVANIA, DEPARTMENT OF MATHEMATICS, 209 SOUTH 33RD STREET, PHILADELPHIA, PA 19104, kaitian@sas.upenn.edu

ROBIN PEMANTLE, UNIVERSITY OF PENNSYLVANIA, DEPARTMENT OF MATHEMATICS, 209 SOUTH 33RD STREET, PHILADELPHIA, PA 19104, pemantle@math.upenn.edu

*Subject classification:* 05A16; secondary 57Q99.

*Keywords*: analytic combinatorics, generating function, algebraic, coefficient extraction.

---

[1] Authors, affiliations, addresses, acknowledgements of support

# 1 Introduction

Analytic combinatorics in several variables (ACSV) seeks to extract the asymptotic behavior of coefficients from a multivariate generating function via analytic means. If $F(\mathbf{z}) = \sum_{\mathbf{r} \in \mathbb{Z}^d} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ converges in some domain to an analytic function, then the coefficient $a_{\mathbf{r}}$ may be exactly recovered via the multivariate Cauchy integral formula

$$a_{\mathbf{r}} = \left( \frac{1}{2\pi i} \right)^d \int_T \mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}} \tag{1.1}$$

where $T$ is a torus in the open domain of convergence in $\mathbb{C}^d$ of the series for $F$ and $d\mathbf{z}/\mathbf{z}$ is the logarithmic volume form $dz_1 \wedge \cdots \wedge dz_d / \prod_{j=1}^d z_j$.

It is usually not feasible to compute this integral exactly, but good approximations are often possible. Asymptotic formulas for $a_{\mathbf{r}}$ are usually valid as $\mathbf{r} \to \infty$ in a certain **direction**, meaning that $\mathbf{r}/|\mathbf{r}|$ converges to a given unit vector $\hat{\mathbf{r}}$ or equivalently that $\mathbf{r}$ converges in projective space; thus directions are specified by either unit vectors in the nonnegative orthant or by projective vectors $[r_1 : \cdots : r_d]$. If $F(\mathbf{z}) = P(\mathbf{z})/Q(\mathbf{z})$ is a rational function and the complex algebraic hypersurface $\mathcal{V} := \{\mathbf{z} : Q(\mathbf{z}) = 0\}$ is smooth, then there is a finite set $E$ of points on $\mathcal{V}$ such that the coefficients $C_{\ell, \mathbf{w}}$ of an asymptotic expansion

$$a_{\mathbf{r}} \approx \sum_{\mathbf{w} \in E} \mathbf{w}^{-\mathbf{r}} \sum_{\ell=0}^{\infty} C_{\mathbf{w}, \ell} |\mathbf{r}|^{-d/2 - \ell} \tag{1.2}$$

may be computed automatically, subject to some further and sometimes more difficult verification of which finite subset $E$ of the finite set of **critical points** of $\mathcal{V}$ should be summed over. The set $E$, and the constants $C_{\mathbf{w}, \ell}$ depend only on $[\mathbf{r}]$ and the approximations are uniform as $[\mathbf{r}]$ varies over some compact neighborhood.

If the denominator of the rational function $F$ defines a variety with singularities, the situation is more complicated but still, in many cases, formulas analogous to (1.2) can be proved. The smooth case was first analyzed in [PW02], with other cases following in [PW04, RW07, BP11, BMP19]. The method is the subject of several books [PW13, Mis19, Mel21].

A complexity hierarchy for functions with finite closed form specifications begins with rational functions, then extends to algebraic functions and so-called D-finite or differentially finite functions. It is known [Fur67, Saf00] that the coefficient array of any algebraic function appears as a diagonally embedded sub-array of the coefficients of some rational function in one more variable. This may be used to reduce the problem of coefficient extraction for algebraic functions to the same problem for rational functions. The applicability of the embedding result to ACSV was first noticed by Raichev and Wilson in [RW07, RW02] and is exploited in [GMRW22] to compute coefficients asymptotics for a number of algebraic generating functions of combinatorial interest. The steps of their **diagonal embedding method** are as follows.

1. Use theorems of Furstenburg or Safonov to embed the coefficient array of an arbitrary algebraic function $F$ diagonally in the coefficient array of rational function $\tilde{F}$.

2. Apply the multivariate Cauchy integral formula.

3. Transfer the integral to the integral of a residue form over an intersection cycle in the pole variety.

4. Use Morse theoretic techniques to represent the intersection cycle as a sum of cycles local to critical points of the height function.

5. Evaluate each of these integrals asymptotically via stationary phase integration methods.

The purpose of the present note is to give an alternative method for algebraic generating functions that avoids some of the complexities of the Raichev-Wilson method. The primary reason for doing so is not to streamline the computation, although we do provide specialized formulas for stationary phase asymptotics that simplify the most common computations arising in coefficient extraction for algebraic series. The chief motive for developing the alternative **lifting method** is transparency. Its derivation relies only on stationary phase methods, and does not use residue forms or intersection cycles. It also avoids the use of a black-box embedding in Step 1 of the diagonal embedding approach. The lifting method is therefore considerably more accessible to the analytic combinatorics community.

## 2   Integral representations

### 2.1   Notation

To specify an algebraic generating function, one requires a defining polynomial along with a choice of solution near the origin. Some global notation is as follows. Fix an integer $d \geq 1$. The coordinates of $\mathbb{C}^{d+1}$ will be denoted $z_1, \ldots, z_d, f$. The $(d+1)$-coordinate plays a different role from the others. Accordingly, we let $\pi : \mathbb{C}^{d+1} \to \mathbb{C}^d$ denote projection to the first $d$ coordinates, and we make the roles of the $z$ variables visually easier to distinguish from $f$ by denoting $(z_1, \ldots, z_d)$ by $\mathbf{z}$ and $(z_1, \ldots, z_d, f)$ by $(\mathbf{z}, f)$. Names of objects in $\mathbb{C}^{d+1}$ will typically have a tilde, while names of their projections to $\mathbb{C}^d$ will drop the tilde.

We assume throughout that $P = \sum_{j=0}^m p_j(\mathbf{z}) f^j$ is a real polynomial function on $\mathbb{C}^{d+1}$, written as a polynomial in $\mathbb{R}[\mathbf{z}][f]$. Suppose there is a neighborhood $\mathcal{N}$ of the origin in $\mathbb{C}^d$ on which there is an absolutely convergent power series $F(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ satisfying $P(\mathbf{z}, F(\mathbf{z})) = 0$. Because there are at most $m$ locally analytic solutions to $P(\mathbf{z}, F) = 0$, one of these may be uniquely specified by naming $P$ together with the values of $F$ at a set of points of cardinality at most $m$. We assume throughout that $P(\mathbf{0}, \cdot)$ has a simple root at a real value $f_0$, and that the $d$-variable algebraic generating function $F$ has been specified by $P(\mathbf{z}, F) = 0$ and $F(\mathbf{0}) = f_0$. Assume without loss of generality, that $P$ is irreducible and square-free; for if it is not, then replacing $P$ by its square-free part (the generator of the radical of the ideal generated by $P$) defines the same solutions, and some irreducible factor defines $F$.

Let $\widetilde{\mathcal{V}}$ denote the variety $\{P = 0\}$ in $\mathbb{C}^{d+1}$ where the coordinates of $\mathbb{C}^{d+1}$ are $z_1, \ldots, z_d, f$. The map $\mathbf{z} \mapsto (\mathbf{z}, F(\mathbf{z}))$ on the domain of convergence $\mathcal{D}$ of $F$ is inverted by $\pi$. It is smooth on some neighborhood of the origin because a power series is smooth on the interior of its domain of absolute convergence. Therefore, it is a diffeomorphism from such a neighborhood onto its lifting into $\widetilde{\mathcal{V}}$. Letting $T$ be any torus within the domain of absolute convergence of the series for $F$, we denote by $\tilde{T}$ the lifting of $T$ into $\widetilde{\mathcal{V}}$. Figure 1 illustrates these definitions (those in red will be defined later).

The usual methodology of stationary phase integration is to move the contour of integration, $T$, into a position where it passes through a stationary phase point where the gradient of the "large term" $\mathbf{z}^{-\mathbf{r}}$ vanishes. Univariate functions with branch points typically require a customized contour, for example one that hugs a slit (a segment or ray whose removal get rid of the branching) at a distance going to zero. A multivariate
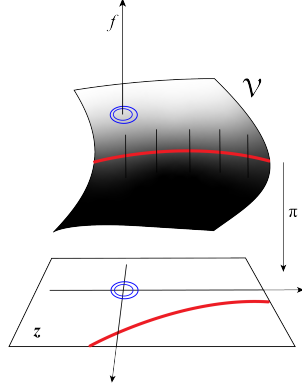
3

Figure 1: The variety $\widetilde{\mathcal{V}}$, projection $\pi$, tori $T$ and $\tilde{T}$, branch locus **br** and vertical tangent locus $\widetilde{\textbf{br}}$    {fig:proj}

version of such a contour is not obvious. One case, namely a nonintegral power of a polynomial, was handled in [Gre18]. Our method is distinct from both the branch contour method of [Gre18], and the diagonal method of [RW07], namely we will transfer the integral "upstairs" to $\widetilde{\mathcal{V}}$, where everything becomes smooth.

## 2.2 Integration upstairs

{pr:upstairs}T

**Proposition 2.1.** *The coefficients $a_{\mathbf{r}}$ in the Laurent expansion of $F$ are given by*

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^d \int_{\tilde{T}} \mathbf{z}^{-\mathbf{r}} f \frac{d\mathbf{z}}{\mathbf{z}} \tag{2.1}$$    {eq:eta}

*where $\mathbf{z}^{-\mathbf{r}}$ denotes $\prod_{j=1}^d z_j^{r_j}$ and $d\mathbf{z}/\mathbf{z}$ denotes the logarithmic volume form $\left(\prod_{j=1}^d z_j^{-1}\right) dz_1 \wedge \cdots \wedge dz_d$.*

PROOF:   Because $\pi^* d\mathbf{z}/\mathbf{z}$ is still $d\mathbf{z}/\mathbf{z}$ in global coordinates, while $\pi^* F|_T = F \circ \pi|_{\tilde{T}} = f$, we see that

$$\pi^* F(\mathbf{z}) \mathbf{z}^{-\mathbf{r}} \frac{d\mathbf{z}}{\mathbf{z}} = \mathbf{z}^{-\mathbf{r}} f \frac{d\mathbf{z}}{\mathbf{z}} \, .$$

Functoriality then implies that the RHS of (2.1) is equal to the RHS of (1.1), proving the proposition.    □

Whereas $F$ may be defined only in a small domain, not extendable around branchpoints or through poles, the form on the RHS of (2.1), which we denote $\eta := \mathbf{z}^{-\mathbf{r}} f d\mathbf{z}/\mathbf{z}$, is well defined and holomorphic on all of $\mathbb{C}_*^{d+1}$. Here, $\mathbb{C}_* := \mathbb{C} \setminus \{0\}$ denotes the set of nonzero complex numbers[2]. We cannot deform $\tilde{T}$ freely through $\mathbb{C}_*^{d+1}$ without altering the integral, but we can deform it within $\widetilde{\mathcal{V}}$, or any other complex $d$-manifold in which it happens to reside.

{pr:can slide}

**Proposition 2.2.** *If $\tilde{T}$ and $\tilde{T}'$ are homotopic within $\widetilde{\mathcal{V}}$ then $\int_{\tilde{T}} \eta = \int_{\tilde{T}'} \eta$.*

PROOF:   Let $H : \tilde{T} \times [0,1] \to \mathcal{V}$ be such a homotopy. From Stokes' Theorem we see that

$$\int_H d\eta = \int_{\partial H} \eta = \int_{\tilde{T}'} \eta - \int_{\tilde{T}} \eta \, .$$

---

[2]We use a subscript of $*$ instead of the more conventional superscript in order to avoid a double superscript.

The integral of the holomorphic $(d+1)$-form $d\eta$ vanishes over any chain supported in a complex $d$-manifold. Because $H$ is a homotopy within $\widetilde{\mathcal{V}}$, the integral of the left-hand side vanishes, proving the proposition. $\quad\square$

## 2.3 Stationary phase integration

The integrand in (2.1) is easy to evaluate asymptotically when it has the form of a **stationary phase** integral. It is a little easier to see why (2.1) is a stationary phase integral if we let $N := |\mathbf{r}| = \sum_{j=1}^{d} r_j$, let $\hat{\mathbf{r}} := N^{-1}\mathbf{r}$, and write the integrand as

$$I(N) = I(N; A, \phi) := A(\mathbf{z})\exp(-N\phi(\mathbf{z}))\, d\mathbf{z} \qquad (2.2) \quad \texttt{\{eq:amp phase\}}$$

where $A(\mathbf{z}) := f/\prod_{j=1}^{d} z_j$ and $\phi(\mathbf{z}) := \sum r_j \log z_j$. In an integral of such a form, the term $A(\mathbf{z})\, d\mathbf{z}$ is called the **amplitude** and $\phi$ is called the **phase**. We have used $N$ rather than the traditional $\lambda$ for the parameter that goes to infinity to remind us that $N\hat{\mathbf{r}}$ is always an integer vector so $\exp(-N\phi(\mathbf{z})) = \mathbf{z}^{-\mathbf{r}}$ and the branching of the logarithm does not matter. The following defines critical points at smooth points of algebraic varieties. There is a more general definition of stratified critical points that need not concern us here.

Define the **logarithmic gradient** of an analytic function $g$ to be the vector whose coordinates are the partial derivatives in logarithmic coordinates:

$$\nabla_{\log} g := \left( z_j \frac{\partial g}{\partial z_j} \right)_{1 \le j \le d}. \qquad (2.3) \quad \begin{array}{l}\texttt{\{eq:loggrad\}}\\ \texttt{\{def:crit\}}\end{array}$$

**Definition 2.3** (critical points and directions)**.**

(i) *A **(smooth) critical point** for a function $\phi$ on an algebraic variety $\mathcal{M}$ is a point $\mathbf{p}$ which is a smooth point of $\mathcal{M}$ and satisfies $d\phi|_{\mathcal{M}}(\mathbf{p}) = 0$.*

(ii) *A **critical point in direction** $\hat{\mathbf{r}}$ on an algebraic variety $\mathcal{M}$ is a smooth critical point for the function $\phi|_{\hat{\mathbf{r}}}$ where $\phi_{\hat{\mathbf{r}}}(\mathbf{z}) := \sum_{j=1}^{d} \hat{r}_j \log z_j$.*

Smooth critical points on the surface $\{P = 0\}$ in direction $[\mathbf{r}]$ satisfy $\nabla_{\log} P(\mathbf{z}) = [\mathbf{r} : 0]$, projectively. This may be captured by the $d+1$ **critical point equations** (see [PWM23, equation (7.8)] or [Mel21, page 11]):

$$\begin{aligned} P(\mathbf{z}) &= 0 \\ \frac{\partial P}{\partial f}(\mathbf{z}) &= 0 \\ r_j z_1 \frac{\partial P}{\partial z_1}(\mathbf{z}) - r_1 z_j \frac{\partial P}{\partial z_j}(\mathbf{z}) &= 0 \qquad 2 \le j \le d\,. \end{aligned} \qquad (2.4) \quad \texttt{\{eq:CPE\}}$$

Generically, this defines a finite set and is easily computed by a computer algebra system. When $\widetilde{\mathcal{V}}$ is smooth, which will usually be the case, these equations precisely define the set of critical points[3].

---

[3]Where $\widetilde{\mathcal{V}}$ is not smooth, i.e., where $\nabla P$ vanishes, the last $d-1$ equations (2.4) are trivially satisfied and one requires further equations for criticality; this will not concern us, as we assume smoothness.

**Definition 2.4** (stationary phase). *Suppose a contour $\Gamma$ contains finitely many critical points for $\phi$. Let* `contrib` *denote the subset of these at which $\Re\{\phi\}$ achieves its minimum on $\Gamma$. Points of* `contrib` *are called* ***stationary phase points*** *for $\phi$ on $\Gamma$, and if* `contrib` *is nonempty, $\Gamma$ is said to be in* ***stationary phase position***.

We remark that being a critical point on $\Gamma$ is in principle a weaker condition than being a critical point on a variety $\mathcal{M}$ that $\Gamma$ lies in, however it is the same wherever the real tangent space to $\Gamma$ has the same span over $\mathbb{C}$ as the tangent space to $\mathcal{M}$; this will always be the case for our contours. Existence of stationary phase points is what makes an integral of the form $\int_\Gamma A e^{-N\phi}$ easy to evaluate asymptotically. The precise nature of $\Gamma$ is not relevant, only the orientation of $\Gamma$, along with the fact that $\mathbf{p}$ is a critical point at which $\Re\{\phi\}$ is minimized on $\Gamma$. Off-the-shelf stationary phase computations at this level of generality can be found in [PWM23, Lemma 5.15], [MW19, Section 7.1], or [PV19, Theorem 4.2]. We find it useful to state coordinate-free hypotheses when possible, while giving the resulting formulae in coordinates. For example in Proposition 2.6, the input data are a phase function $\phi$ on a complex $d$-manifold and a holomorphic $d$-form $\eta$ for the amplitude, while the formula for the integral uses a coordinate representation $A(\mathbf{z})d\mathbf{z}$ for $\eta$.

All the examples in this paper have an expansion (1.2) in which all terms with $\ell = 0$ vanish. We therefore find it convenient to state an explicit formula for the leading term $C_1$, in the special case that the amplitude $\eta = \mathbf{z}^{-\mathbf{r}} f\, d\mathbf{z}/\mathbf{z}$ vanishes to order precisely 1 at the stationary phase point. We base our formulae on some useful reductions for this case that can be found in the Appendix to [MW19].

**Definition 2.5.** *The notation $\sqrt{\det M}$ denotes the product of the principal square roots of the eigenvalues of the matrix $M$. The notation $(\det M)^{1/2}$ leaves open which choice of square root is intended.*

We begin with the case where `contrib` is a singleton $\{\mathbf{p}\}$. It is well known that the leading term of a stationary phase integral is inversely proportional to a curvature invariant at $\mathbf{p}$, which is given in coordinates by the determinant of the Hessian matrix of the phase function. These formulae make more sense when one takes into account the way such a determinant transforms under changes of variable. If the Jacobian is $J$ then at a point where the gradient vanishes, the Hessian matrix $H$ transforms to $J^T H J$; as the amplitude $A(\mathbf{z})d\mathbf{z}$ transforms to $\det(J)A(\mathbf{z}')d\mathbf{z}'$, this means that

$$\frac{A}{\det(H)^{1/2}} \quad \text{is independent of the choice of coordinates.} \tag{2.5}$$

**Proposition 2.6** (stationary phase formula and case where amplitude vanishes to order 1). *Let $\eta$ be a homolorphic $d$-form on a complex $d$-manifold $\mathcal{M}$ and let $\phi$ be a holomorphic function on $\mathcal{M}$. Let $\mathbf{p}$ be a point of $\mathcal{M}$ which $d\phi$ vanishes. Fix a coordinate system $z_1, \ldots, z_d$ on a neighborhood of $\mathbf{p}$ in $\mathcal{M}$ and suppose*

  *(i) the form $\eta$ is represented by $A(\mathbf{z})d\mathbf{z}$;*

  *(ii) the function $\phi$ has a nondegenerate Hessian matrix $H$ at $\mathbf{p}$, which condition is invariant under coordinate changes[4] by (2.5).*

*Define quantities*

---

[4]Technically, a coordinate system is a map $\Psi(\mathcal{M}, \mathbf{p}) \to (\mathbb{C}^d, \mathbf{0})$ and we should refer to the Hessian matrix of $\phi \circ \Psi^{-1}$ at $\mathbf{0}$, however continue to use "$\phi(\mathbf{p})$ in coordinates", "$A(\mathbf{p})$ in coordinates" and so forth instead of $\phi \circ \Psi^{-1}(\mathbf{0})$, $(\Psi^{-1})^* A(\mathbf{0})$ and so forth because most readers can more easily read the former.

- $g(\mathbf{z}) := \phi(\mathbf{z}) - \frac{1}{2}(\mathbf{z} - \mathbf{p})^T H(\mathbf{z} - \mathbf{p})$, *in other words, $\phi$ with its leading (quadratic) term subtracted off;*

- *a second order differential operator* $\mathbb{H} := \displaystyle\sum_{i,j=1}^{d} -(H^{-1})_{ij}\frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j}$ .

*Let $\mathcal{N}$ be a neighborhood of the origin in $\mathbb{R}^d$ and let $\Gamma : (\mathcal{N}, \mathbf{0}) \to (\mathcal{M}, \mathbf{p})$ be a compact, smooth, real $d$-chain supported on a set $|\Gamma|$ on which $\Re\{\phi\}$ is uniquely minimized at an interior point, $\mathbf{p}$. Then the integral*

$$I(N) := \int_\Gamma \eta \exp(-N\phi) \qquad (2.6) \quad \{\texttt{eq:int}\}$$

*has an asymptotic series expansion*

$$I(N) \approx e^{-N\phi(\mathbf{p})}\left(\frac{2\pi}{N}\right)^{d/2}\sum_{\ell=0}^{\infty} C_\ell N^{-\ell} \qquad (2.7) \quad \{\texttt{eq:I}\}$$

*for some constants $\{C_\ell\}$ that can be computed from the partial derivatives of $\phi$ and $A$ at $\mathbf{p}$. Specifically,*

$$C_\ell = (\det H)^{-1/2}\sum_{j=0}^{2\ell}(-1)^\ell \frac{\mathbb{H}^{\ell+j}(A \cdot g)}{2^{\ell+j}j!(\ell+j)!}(\mathbf{p}). \qquad (2.8) \quad \{\texttt{eq:C\_l}\}$$

*If $A(\mathbf{p}) = 0$ and $dA(\mathbf{p}) \neq \mathbf{0}$, then $C_0 = 0$ and the leading term is given by*

$$C_1 = -\frac{1}{2(\det H)^{1/2}}\left[\mathbb{H}(A)(\mathbf{p}) + \frac{1}{4}\mathbb{H}^2(A \cdot g)(\mathbf{p})\right]. \qquad (2.9) \quad \{\texttt{eq:degree 3}\}$$

*The square root in (2.9) should be chosen as follows. In the coordinate system that represents $\eta = A\,d\mathbf{z}$, the chain $\Gamma : \mathbb{R}^d \to \mathcal{M}$ pulls back to a chain $\gamma : \mathbb{R}^d \to \mathbb{C}^d$; let $J = d\mathbf{z}/d\mathbf{t}$ denote the Jacobian matrix of $\gamma$; choose the $\frac{1}{2}$ power in (2.9) to be $\det J/\sqrt{\det J^T H J}$, which is a choice for $\det H^{1/2}$ by (2.5).*

PROOF: Lemma A.3 and Theorem 5.5 of [MW19] compute a general asymptotic series expansion for power series coefficients, first writing this as $(2\pi i)^{-d}\int \tilde{u}(\mathbf{z})\exp(\tilde{g}(\mathbf{z}))d\mathbf{z}$ and then evaluating this integral asymptotically. The form of their series is as given in (2.7) – (2.8).

When $A(\mathbf{p}) = 0$, $C_0$ vanishes. When $dA(\mathbf{p}) \neq 0$, $C_1$ does not vanish. Plugging in $\phi$ for $\tilde{g}$ and $A$ for $\tilde{u}$, their result is that our $I(N)$ is asymptotic to

$$L_1(A, \phi)\, N^{-1}\left(\frac{2\pi}{N}\right)^{d/2}(\det H)^{-1/2}$$

where $L_1(A, \phi)$ is given by several formulas, of which we use the second displayed equation in their Lemma A.3 due to our assumption that $A$ vanishes to order precisely 1. Equation (2.9) is then a direct statement of their result.

To check the choice of sign, first note that $\det J/\sqrt{\det J^T H J}$ is indeed a square root of $\det H$. Pulling back the integral $I(N) = \int_\Gamma \eta \exp(-N\phi)$ to $\int_\gamma A(\mathbf{z}(\mathbf{t}))J\,d\mathbf{t}\exp(-N\phi \circ \Gamma^{-1})$ and applying formula (5.4) of [PWM23] shows that this sign choice evaluates the integral (see also [PWM23, equation (5.6)]). □

Section 5 computes a number of examples, mostly taken from [GMRW22], where the majority have $d = 2$ (bivariate algebraic generating functions). It is helpful to pre-compute (2.9) for $d = 1, 2$: not only does this show the degree of simplification but it gives users an off-the-shelf formula that does not require them to program a differential operator in their computer algebra platform. Section 5.2 contains some further symbolic algebra techniques for obtaining simplified representations for algebraic quantities such $C_1$. When $d = 1$, the Hessian matrix $H$ reduces to the scalar quantity $V := \phi''(\mathbf{p})$, while the operator $\mathbb{H}$ is $V^{-1}$ times the second derivative operator. This leads to a rather compact formula. While the explicit formula for $d = 2$ is somewhat messier, we will see in Section 5.2 that the formula can simplify drastically when some of the partial derivatives vanish.

{cor:d = 1,2}

**Corollary 2.7.** *When $d = 1$, the formula* (2.9) *reduces to the following expresion.*

$$C_1 = -\frac{1}{2} \cdot V^{-1/2} \left[ -\frac{1}{V} A''(\mathbf{p}) + \frac{1}{V^2} \phi'''(\mathbf{p}) A'(\mathbf{p}) \right] \qquad (2.10) \quad \{\text{eq:d=1}\}$$

*When $d = 2$, the formula* (2.9) *reduces to the following expression, where again $H$ denotes $\phi_{xx}\phi_{yy} - \phi_{xy}^2$ and all partial derivatives are evaluated at $\mathbf{p} = (p_1, p_2)$.*

$$
\begin{aligned}
C_1 \;=\; & -\frac{1}{2} H^{-1/2} \times \\
& \Big[ H^{-1} \left( -A_{xx}\phi_{yy} + 2A_{xy}\phi_{xy} - A_{yy}\phi_{xx} \right) \\
& + H^{-2} \Big( A_x \phi_{yy}^2 \phi_{xxx} - A_y \phi_{xxx}\phi_{xy}\phi_{yy} \\
& - 3A_x \phi_{xxy}\phi_{xy}\phi_{yy} + (A_x\phi_{xyy} + A_y\phi_{xxy})(\phi_{xx}\phi_{yy} + 2\phi_{xy}^2) \\
& - 3A_y \phi_{xyy}\phi_{xy}\phi_{xx} - A_x \phi_{yyy}\phi_{xy}\phi_{xx} + A_y \phi_{xx}^2 \phi_{yyy} \Big) \Big]
\end{aligned}
\qquad (2.11) \quad \{\text{eq:d=2}\}
$$

*The square root in both cases is chosen as in Proposition 2.6.*

Proposition 2.6 and Corollary 2.7 extend easily to allow `contrib` to be a finite set of cardinality greater than 1. The following generalization can be found in [PWM23, Theorem 5.3; see also Theorem 9.25].

{cor:finite su

**Corollary 2.8.** *Proposition 2.6 and Corollary 2.7 continue to hold if the hypothesis of a single critical point $\mathbf{p}$ at which $\Re\{\phi\}$ is uniquely minimized is replaced by the hypothesis that there are finitely many critical points $\mathbf{p}^{(1)}, \ldots, \mathbf{p}^{(m)}$ of $\phi$ on $\Gamma$ whose common value of $\Re\{\phi\}$ attains the minimum on $\Gamma$, and the conclusion* (2.7) *is altered replacing the right-hand side by a sum of the same quantity with $\mathbf{p}$ replaced by $\mathbf{p}^{(j)}$, for $j = 1, \ldots, m$.* □

## 2.4 The lifting method

{ss:lifting}

Putting together Propositions 2.1, 2.2 and the computation in Proposition 2.6 we obtain the following plan for computing coefficient asymptotics in a direction $[\mathbf{r}]$ for the series $F$.

(i) Compute the set of critical points on $\widetilde{\mathcal{V}}$.

(ii) Deform $\tilde{T}$ to a contour $\tilde{T}'$ in stationary phase position, so that it there is a nonempty finite set `contrib` of stationary phase points for $\phi$ on $\tilde{T}'$ at which $\Re\{\phi\}$ takes the value $\inf_{\mathbf{z} \in \tilde{T}'} \Re\{\phi(\mathbf{z})\}$.

(iii) Use standard stationary phase estimates to asymptotically evaluate $\int_{\tilde{T}'} A(\mathbf{z}) \exp(-N\phi(\mathbf{z})) \, d\mathbf{z}$.

The first and last of these steps require viturally no work, the first being accomplished by (2.4) and Proposition 4.1 and the last by Proposition 2.6 and Corollary 2.8. Already, with these two steps accomplished, one can get a pretty good idea of the asymptotics of $a_{\mathbf{r}}$. Each critical point $\mathbf{p}$ leads to an asymptotic series of the form (2.7); according to Corollary 2.8, summing these over the correct set `contrib` of critical points for $\phi$ on $\widetilde{\mathcal{V}}$, with the correct orientations, will give an asymptotic series for $a_{\mathbf{r}}$, provided the torus $\tilde{T}$ may be deformed into stationary phase position.

Often it is easy to determine by inspection which among finitely many choices for `contrib` yield the correct asymptotics. There are also cases where the homotopy mapping taking $\tilde{T}$ to an appropriate chain $\tilde{T}'$ is geometrically obvious. For the remainder of the cases, we develop a number of sufficient conditions for determining `contrib`. While rigorous numerical homotopy procedures exist that may be used to investigate deformations of $\tilde{T}$ into stationary phase position, these are cumbersome and few researchers possess or are familiar with the software. Lemma 4.4 gives a method to compute `contrib` without resorting to numerical homotopies. The remainder of this section contains a preliminary discussion of the properties of the covering map $\pi$, along with various assumptions that hold in many applications and greatly simplify the identification of `contrib`.

One way to deform $\tilde{T}$ into stationary phase position is to enlarge the first $d$ coordinates so as to remain a torus at every fixed time, while varying $f$ so that $(\mathbf{z}, f)$ remains in $\widetilde{\mathcal{V}}$. By the nature of $\phi$, this ensures that $\Re\{\phi\}$ is constant on the resulting torus, which means that at the end of the homotopy, the final contour $\tilde{T}'$ will be in stationary phase position, provided the set of critical points on $\tilde{T}'$ is finite and nonempty. This type of deformation can be described as a homotopy of $T$, lifted via $\pi^{-1}$ to $\widetilde{\mathcal{V}}$. Computationally, it is easier keep track of deformations of $T$ in $\mathbb{C}^d$ than deformations of $\tilde{T}$ in $\widetilde{\mathcal{V}}$. Therefore, we have two reasons to pursue deformations that can be described as liftings via $\pi^{-1}$ of homotopies in $\mathbb{C}^d$.

{def:discr}

**Definition 2.9.** *The **discriminant** $\mathrm{discr}(P)$ of $P$ with respect to $f$ is the minimal polynomial in $\mathbf{z}$ that vanishes if and only if $P(\mathbf{z}, \cdot)$ does not have distinct roots[5]. Let $\mathbf{br} \subseteq \mathbb{C}^d$ denote the branching locus, that is, the algebraic hypersurface defined by $\mathrm{discr}(P)$. The **vertical tangent locus** $\widetilde{\mathbf{br}}$ is the set of points $(\mathbf{z}, f) \in \widetilde{\mathcal{V}}$ such that $\partial P / \partial f = 0$. Under the assumption that $\widetilde{\mathcal{V}}$ is smooth, this is precisely the set of points of $\widetilde{\mathcal{V}}$ whose tangent space is vertical. A lower star, such as $\mathbf{br}_*, \widetilde{\mathbf{br}}_*, \widetilde{\mathcal{V}}_*$, refers respectively to not allowing a zero in the first $d$ coordinates.*

**Proposition 2.10.** *The image of $\widetilde{\mathbf{br}}$ under $\pi$ is $\mathbf{br}$. The polynomial $\mathrm{discr}(P)$ may be computed by eliminating $f$ from the ideal generated by $P$ and $\partial P / \partial f$.*

PROOF: The first statement, namely that vertical tangents occur precisely where roots coalesce, is well known. The second follows from the fact that projection to the $\mathbf{z}$-plane corresponds to eliminating $f$. $\square$

# 3 Main results

{sec:main}

To reiterate standing assumptions on $P$ and $F$, we assume:

---

[5]One needs to be careful how one categorizes those $\mathbf{z}$ for which one of the roots goes to infinity, but that won't be relevant for us.

- $P$ is a smooth real polynomial on $\mathbb{C}^{d+1}$ whose zero set is denoted $\widetilde{\mathcal{V}}$.

- All roots of $P(\mathbf{0}, \cdot)$ are simple and $\mathrm{discr}(P)$ is squarefree.

- $F$ is an algebraic power series in a neighborhood of the origin in $\mathbb{C}^d$ defined by the equation $P(\mathbf{z}, F(\mathbf{z})) = 0$ and the initial condition $F(\mathbf{0}) = f_0$.

**Definition 3.1** (rank)**.** *Let $\xi_1 \geq \cdots \geq \xi_k$ denote the finite real roots of $P(\mathbf{0}, \cdot)$ listed in decreasing order. By convention we consider $P$ to have $\ell$ real roots at $+\infty$ if for all $M > 0$, all vectors $\mathbf{v}$ of positive real numbers, and sufficiently small $\varepsilon = \varepsilon(M) > 0$, the univariate polynomial $P(\varepsilon\mathbf{v}, \cdot)$ has $\ell$ positive real roots greater than $M$. If $f_0 = \xi_j$, we say the **rank** $\mathrm{rk}(\mathrm{f}_0)$ of $f_0$ is defined to be $\ell + j$.*

**Definition 3.2** (minimal point)**.** *Let $T$ be the centered torus containing a point $\mathbf{z} \in \mathbb{C}_*^d$ Say that $\mathbf{z}$ is **weakly minimal** for $G$ if $G$ is nonvanishing on the torus $t \cdot T$ for all $0 < t < 1$. Weak minimality is implied by the usual notion of **minimality** in ACSV [Mel21, Definition 3.9], namely that $G(\mathbf{w}) \neq 0$ for every $\mathbf{w}$ satisfying $|w_j| \leq |z_j|$ for all $j$, with at least one of the inequalties being strict.*

For the next definition, all we need is a real polynomial $P$, a real number $f_0$ such that $P(\mathbf{0}, f_0) = 0$, and the assumption that all roots of $P(\mathbf{0}, \cdot)$ are simple.

**Definition 3.3.** *Let $(\mathbf{z}, f_1)$ be a point of $\widetilde{\mathcal{V}}$ and let $T'$ denote the torus through $\mathbf{z}$. Say that $(\mathbf{z}, f_1)$ is **on the branch defined** by $f_0$ if the map $H : T' \times [0, 1] \to \mathbb{C}^d$ by $H(\mathbf{w}, t) = t\mathbf{w}$ lifts to a continuous map $\tilde{H} : T' \times [0, 1] \to \widetilde{\mathcal{V}}$ with $\tilde{H}(\mathbf{w}, 0) = f_0$ for every $\mathbf{w} \in T'$ and $\tilde{H}(\mathbf{z}, 1) = f_1$.*

The following is our main result and will be proved in Section 4.                    {th:main}

**Theorem 3.4.** *Let $\mathbf{z} := \exp(\mathbf{x}) \in \mathbf{br}$ be a positive real zero of $\mathrm{discr}(P)$. Assume $\mathbf{z}$ is a smooth, critical point for $\mathrm{discr}(P)$ in direction $[\mathbf{r}]$, with $p_m(\mathbf{z}) \neq 0$ and $\mathbf{z}$ weakly minimal for $p_m \cdot \mathrm{discr}(P)$. Let $(\mathbf{z}, f_1) \in \widetilde{\mathbf{br}}$ be a real point of the vertical tangent locus.*

**Conclusion 1:**

> *The point $(\mathbf{z}, f_1)$ is critical for $P$ in direction $[\mathbf{r} : 0]$. It is on the branch defined by $f_0$ if and only if $f_1 = \xi_{\mathrm{rk}(\mathrm{f}_0)}$ where $\xi_1 \geq \xi_2 \geq \ldots$ enumerates the real roots of $P(\mathbf{z}, \cdot)$ in decreasing order with multiplicities.*

*Assuming that $(\mathbf{z}, f_1)$ is on the branch defined by $f_0$, suppose the set of $\mathbf{y}$ such that there is a complex number $f_{\mathbf{y}}$ with $\exp(\mathbf{x} + i\mathbf{y}, f_{\mathbf{y}})$ is on the branch of $\widetilde{\mathcal{V}}$ defined by $f_0$ and critical in direction $[\mathbf{r}]$ is finite and denote this set by $W$. Finally, assume for each $\mathbf{y} \in W$, the root $f_{\mathbf{y}}$ of $P(\exp(\mathbf{x} + i\mathbf{y}), \cdot)$ has multiplicity precisely 2 and that the Hessian determinant of $\phi_{\mathbf{r}} := \sum_{j=1}^d r_j \log z_j$ restricted to $\widetilde{\mathcal{V}}$ is nonsingular. Then,*

**Conclusion 2:**

$$a_{\mathbf{r}} \approx \exp(-\mathbf{r} \cdot \mathbf{x}) \sum_{\ell=1}^{\infty} \sum_{\mathbf{y} \in \mathtt{contrib}} C_{\mathbf{y},\ell} \exp(-i\mathbf{r} \cdot \mathbf{y}) |\mathbf{r}|^{-d/2-\ell} \qquad (3.1) \quad \{\texttt{eq:asym}\}$$

*where the constants $C_{\mathbf{y},\ell}$ are the constants $C_\ell$ determined in Proposition 2.6 with $\mathbf{p} = \exp(\mathbf{x} + i\mathbf{y})$, as well as $\phi(\mathbf{z}) = \hat{\mathbf{r}} \cdot \log \mathbf{z}$ and $\eta = f dz_1 \cdots dz_d / \prod_{j=1}^d z_j$. The expansion of (3.1) will be nonzero for some $\ell \geq 1$ and uniform as $[\mathbf{r}]$ varies over compact neighborhoods where the hypotheses hold.*

We amplify on the most common form of the final formula, which occurs in the case that $|W| = 1$. This corresponds, more or less, to the aperiodic case.

**Corollary 3.5** (computational form). *Suppose that $W$ in Theorem 3.4 is a single point $\mathbf{y}$ and denote $\mathbf{p} := \exp(\mathbf{x} + i\mathbf{y})$. Let $\eta := A(\mathbf{z})d\mathbf{z} := f d\mathbf{z}/\prod_{j=1}^{d} z_j$ and let $\phi(\mathbf{z}) := \sum_j \hat{r}_j \log z_j$. Fix $k \leq d$ with $\partial P/\partial z_k$ nonvanishing at $(\mathbf{p}, f_1)$. Then*

$$a_{\mathbf{r}} \sim C|\mathbf{r}|^{-d/2-1}\mathbf{p}^{-\mathbf{r}} \tag{3.2}$$

*where $C$ is the constant $C_{\mathbf{0},1}$ from (3.1) and is determined as follows.*

*Reparametrize $\widetilde{\mathcal{V}}$ by $\{z_j : j \neq k\}$ and $f$. Let $H$ denote the Hessian of $\phi$ in the new coordinates, let $\tilde{\phi}$ represent $\phi$ in the new coordinates with the quadratic term subtracted off, and let*

$$\tilde{A} \, dV_k := A \frac{dz_k}{df} \, dV_k = A \frac{\partial P/\partial f}{\partial P/\partial z_k} \, dV_k \tag{3.3}$$

*denote the form $\eta$ in the new coordinates, where $dV_k := df \wedge dz_1 \wedge \cdots \wedge \widehat{dz_k} \wedge \cdots \wedge dz_d$. Let $\mathbb{H}$ denote the second order differential operator $\mathbb{H} := \sum_{i,j=1}^{d} -(H^{-1})_{ij} \frac{\partial^2}{\partial \tilde{z}_i \partial \tilde{z}_j}$ where $\tilde{z}_j$ denotes $z_j$ if $j \neq k$ and $f$ if $j = k$. Then*

$$C = \left(\frac{1}{2\pi i}\right)^d (2\pi)^{d/2} C_1$$

*with $C_1$ given by (2.9) with $\tilde{A}$ in place of $A$:*

$$C_1 = -\frac{1}{2}(\det H)^{-1/2}\left[\mathbb{H}(\tilde{A})(\mathbf{p}) + \frac{1}{4}\mathbb{H}^2(\tilde{A} \cdot g)(\mathbf{p})\right]. \tag{2.9}$$

*The $\frac{1}{2}$ power should be taken as $i^d/\prod_{j=1}^{d}\sqrt{-N_j}$, where $\{N_j : 1 \leq j \leq d\}$ are the eigenvalues of $H$, and $\sqrt{\phantom{x}}$ denotes the principal square root, namely the one in the right half-plane. Finally, one must multiply by $-1$ if the branch of $F$ at the origin is the larger of the two coalescing at $(\mathbf{p}, f_1)$.*

PROOF: This is a straightforward application of Proposition 2.6 except for the choice of square root. The square root is determined by the oriented tangent plane to the chain of integration near $\mathbf{p}$. In log coordinates, $d\phi$ is positive definite on the real tangent space (by the strong convexity assumption), hence negative definite on the imaginary tangent space. In the original coordinates, at any real point such as $(\mathbf{p}, f_1)$, the imaginary log tangent space maps to the imaginary tangent space. Therefore, the chain $\Gamma$ parametrized by $(f + it_1, p_1 + it_2, \ldots, p_{k-1} + it_k, p_{k+1} + it_{k+1}, \ldots, p_d + it_d)$ has a strict minimum of $\Re\{\phi\}$ at $\mathbf{p}$. The Jacobian determinant of the parametrization is $i^d$ and the Hessian is the negative of $H$, therefore the integral over $\Gamma$ with this parametrization has constant $C_1$ determined by (2.9) with $(\det H)^{1/2}$ taken to be $i^d/\sqrt{\prod_{j=1}^{d}(-N_j)}$.

To go from the integral over $\Gamma$ to the integral over $\tilde{T}'$, observe first that the local homology group has rank 1 at a quadratically nondegenerate critical point, hence $\tilde{T}'$ is homologous either to $\Gamma$ or $-\Gamma$. By Lemma A.1, where $\mathbf{T}_*$ is used for $\tilde{T}'$ and $\mathbf{T}_\varepsilon$ is used for the original small torus $\tilde{T}$, the orientation of $\mathbf{T}_*$ is positive with respect to $df \wedge \eta$ if and only if $f_0$ is the lesser of the two roots eventually coalescing at $(\mathbf{p}, f_1)$. The orientation of $\Gamma$ with respect to $df \wedge \eta$ is positive, hence the integral computed by this parametrization requires a sign flip to compute the integral over $\tilde{T}'$ if and only if $f_0$ is the greater of the two coalescing roots. □

11

**Simplifying conditions**

Theorem 3.4 does not guarantee the existence of such a pair $(\mathbf{z}, f_1)$ satisfying the hypotheses. In fact a number of further hypotheses hold in many examples that help to assure this. We take as our base of examples the twenty examples analyzed in [GMRW22]. Our simplifying assumptions are catalogued in Table 1 along with which examples from [GMRW22] satisfy them. The first is set off because it is a standing hypothesis, repeated here so as to display which examples satisfy it. Four of these do not satisfy our standing hypothesis that $P$ should be smooth; twelve of the twenty satisfy all our simplifying assumptions. All eight of the examples in [GMRW22] that don't satisfy our simplifying hypotheses are among the ten for which no final asymptotic formula given.

| 1 | $P$ is smooth | 16 out 20 examples (all except 1, 5, 6, 20) |
|---|---|---|
| 2 | $p_m$ is a monomial | 13 out of 20 examples (all except 1, 2, 3, 5, 6, 11, 12) |
| 3 | all coefficients of $F$ are nonnegative | 20 out of 20 examples |
| 4 | $P$ is quadratic in $f$ | 18 out of 20 examples (all except 6, 20) |

<div align="center">Table 1: simplifying conditions</div>

{table:1}

The following result will be proved in the next section.                    {th:nice}

**Theorem 3.6.** *In addition to the standing hypotheses, assume all three remaining hypotheses in Table 1. Suppose there is a weakly minimal critical point $(\mathbf{w}, f)$ of $P$ in direction $[\mathbf{r} : 0]$. Define $z_j := |w_j|$. Then the point $\mathbf{z}$ is a minimal critical point of $p_m \cdot \mathrm{discr}(P)$ in direction $[\mathbf{r}]$, where the factor $\mathrm{discr}(P)$ is the one that vanishes; the fiber $\pi^{-1}(\mathbf{z})$ contains a single point $(\mathbf{z}, f_1)$; this point is a minimal critical point for $P$ in direction $[\mathbf{r} : 0]$, and is on the branch determined by $f_0$. Consequently, the conclusions of Theorem 3.4 hold.*

# 4  Proofs and effective procedures

{sec:proofs}

## 4.1  Proof of Theorem 3.4

The critical point equations (2.4) define a (generically) finite subset of $\mathbb{C}^{d+1}$ and may be rewritten as $P(\mathbf{z}) = 0$ together with $\nabla_{\log} P(\mathbf{z}) = [\mathbf{r} : 0]$ in $\mathbb{CP}^d$. The next proposition shows that critical points for $P$ in direction $[\mathbf{r} : 0]$ upstairs correspond to critical points for $\mathrm{discr}(P)$ in direction $[\mathbf{r}]$ downstairs. This establishes the first part of Conclusion 1 of Theorem 3.4.                    {lem:project}

**Lemma 4.1.** *Let $(\mathbf{z}, f)$ be a critical point in direction $[\mathbf{r} : 0]$ for $P$. Then $\mathbf{z}$ is a critical point in direction $[\mathbf{r}]$ for $\mathrm{discr}(P)$. Conversely, if $\mathbf{z}$ is a critical point in direction $[\mathbf{r}]$ for $\mathrm{discr}(P)$ and $(\mathbf{z}, f) \in \widetilde{\mathbf{br}}$, then $(\mathbf{z}, f)$ is critical for $P$ in direction $[\mathbf{r} : 0]$.*

PROOF:    Assume $(\mathbf{z}, f) \in \widetilde{\mathbf{br}}$. The tangent space $T_{(\mathbf{z},f)}(\widetilde{\mathcal{V}})$ is a $d$-dimensional linear space containing the elementary basis vector in the $f$-direction. Consequently, it is mapped by $\pi$ to a $(d-1)$-dimensional subspace of $T_{\mathbf{z}}(\mathbb{C}^d)$. Because $\widetilde{\mathcal{V}} \supseteq \widetilde{\mathbf{br}}$, $T_{(\mathbf{z},f)}(\widetilde{\mathcal{V}}) \supseteq T_{\mathbf{z}}(\widetilde{\mathbf{br}})$, hence

$$\pi(T_{(\mathbf{z},f)}(\widetilde{\mathcal{V}})) \supseteq \pi(T_{(\mathbf{z},f)}(\widetilde{\mathbf{br}})) = T_{\mathbf{z}}(\pi(\widetilde{\mathbf{br}})) = T_{\mathbf{z}}(\mathbf{br}) \,.$$

<div align="center">12</div>

As the first and last linear spaces both have dimension $d - 1$, they must coincide. By hypothesis, $T_{(\mathbf{z},f)}(\widetilde{\mathcal{V}})$ consists of all vectors orthogonal to $[z_1 r_1 : \cdots : z_d r_d : 0]$. Hence $T_{\mathbf{z}}(\mathbf{br})$ consists of all vectors orthogonal to $[z_1 r_1 : \cdots : z_d r_d]$. This establishes the conclusion in both directions. $\qquad\square$

The restriction of $\pi$ to $\widetilde{\mathcal{V}}$ is an $m$-to-1 covering map except over points of two kinds: the branch locus $\mathbf{br}$ and the pole variety $\mathbf{pole}$, defined by the vanishing of the leading coefficient $p_m$. On the pole variety the degree of $P(\mathbf{z}, \cdot)$ is less than $m$, corresponding to one or more roots at infinity. The following proposition states a well known property of algebraic branched coverings; see, e.g., [Hat02, Section 1.3] for further definitions involving covering spaces.

{pr:covering s

**Proposition 4.2.** *Let $\mathcal{A} := \mathbb{C}_*^d \setminus (\mathbf{pole} \cup \mathbf{br})$ and denote $\tilde{\mathcal{A}} := \pi^{-1}(\mathcal{A})$. Then $\pi : \tilde{\mathcal{A}} \to \mathcal{A}$ is an $m$ to 1 covering of $\mathcal{A}$.* $\qquad\square$

Covering spaces are useful because homotopies on the base space lift uniquely to homotopies on the covering space.

{lem:cover}

**Lemma 4.3.** *Let $T$ be a torus in the closure of the domain of convergence of $F$. Let $H : T \times [0,1] \to \mathbb{C}^d$ be a homotopy from $T$ to a chain $T'$ such that $H(\mathbf{z}, t) \in \mathcal{A}$ when $t < 1$ and $H(\mathbf{z}, 1) \notin \mathbf{pole}$. Then there is a unique lifting of $H$ to a homotopy $\tilde{H} : T \times [0,1] \to \widetilde{\mathcal{V}}$ such that $\tilde{H}(\mathbf{z}, 0)$ is the chain $\tilde{T}$. The lifted chain is smooth for every $t < 1$ and continuous at $t = 1$. Consequently,*

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^d \int_{\tilde{T}'} \mathbf{z}^{-\mathbf{r}} f \frac{d\mathbf{z}}{\mathbf{z}}. \qquad (4.1)$$

{eq:minimal}

PROOF: For any $t < 1$, existence and uniqueness of $\tilde{H}|_{[0,t]}$ follow from the homotopy lifting property [Hat02, Proposition 1.30]. When $H(\mathbf{z}, 1) \in \mathcal{A}$, define $\tilde{H}(\mathbf{z}, 1) = \tilde{\gamma}(1)$ where $\tilde{\gamma}$ is the unique lifting of the path $H(\mathbf{z}, t) : 0 \le t \le 1$. When $H(\mathbf{z}, 1) \in \mathbf{br}$, define $\tilde{H}(\mathbf{z}, 1) := \lim_{t \uparrow 1} \tilde{H}(\mathbf{z}, t)$. The limit exists because there is no pole; approaching the branch point, the lift within any branch approaches a limit. Proposition 2.2, along with the identity (2.1) then establishes (4.1). $\qquad\square$

Continuing with the proof of Theorem 3.4, let $T$ be the torus containing $\mathbf{z}$. For every $t < 1$, the torus $t \cdot T$ is in the domain of convergence of $F$ because the only possible singularities of an algebraic function are at poles and branchpoints, and the open polydisk on whose boundary $T$ lies has no poles or branchpoints by the assumption that $\mathbf{z}$ is minimal for $p_m \cdot \text{discr}(P)$. Applying Lemma 4.3 establishes that $(\mathbf{z}, f)$ is on the branch defined by $f_0$ for some $f$ satisfying $P(\mathbf{z}, f) = 0$. To finish proving Conclusion 1 we need to see that choosing $f = \xi_{\text{rk}(f_0)}$ yields the correct branch. This is accomplished by the following lemma, which in one variable goes by the name of the Algebraic Continuation Algorithm [FS09, Proposition VII.8]; the multivariate argument is nearly identical; see [Cha02] for a prior mention and [Mel21, Example 2.16] for a development in the case where the coefficients of $F$ are known to be nonnegative.

{lem:no change

**Lemma 4.4.** *Suppose the line segment $\gamma := \{t\mathbf{v} : 0 < t < 1\}$ lifts to a path avoiding $\widetilde{\mathbf{br}}$, satisfying $\tilde{\gamma}(0) = f_0$ and $\tilde{\gamma}(1) = f_1$. Then, listing the positive real roots of $P(\mathbf{v}, \cdot)$ in decreasing order with multiplicities, $f_1$ will occur at index $\text{rk}(f_0)$.*

PROOF: The homotopy lifting property guarantees a unique lifting because the path $\gamma$ remains in $\mathcal{A}$ up to time 1 and is not in $\mathbf{pole}$ at time 1. Without coalescing, real roots of a continuously varying family of real univariate polynomials cannot become complex nor can complex roots become real. Therefore the real roots of $P(t\mathbf{v}, \cdot)$ along $\gamma$ remain in the same order. By hypothesis they remain finite. Coalescing can occur at

the end, but counting with multiplicity preserves the order. By continuity and by definition of how infinite roots are handled, for sufficiently small $\varepsilon > 0$, $\tilde{\gamma}(\varepsilon)$ will be the real root of $P(\varepsilon \mathbf{v}, \cdot)$ of index $f_0$, hence this will persist up to time $\tilde{\gamma}(1)$. By continuity, this persists at 1 if one counts multiplicities. $\qquad \square$

PROOF OF REMAINING CONCLUSION IN THEOREM 3.4: By definition, the fact that $(\mathbf{z}, f_1)$ is on the branch defined by $f_0$ means that the lift $\tilde{T}$ of a small torus is homotopic to a torus $\tilde{T}'$ through $(\mathbf{z}, f_1)$. By Lemma 4.3, we have the formula (4.1) for $a_{\mathbf{r}}$. Having assumed that $\tilde{T}'$ passes through finitely many critical points for $\phi$ on $\widetilde{\mathcal{V}}$, we see that $\tilde{T}'$ is in stationary phase position for $\phi$. Proposition 2.6 then implies Conclusion 2, provided we verify that $A(\mathbf{z}, f_1) = 0$. But the form $\eta$ necessarily vanishes at the point $\exp(\mathbf{x} + i\mathbf{y})$ because $dz_1 \wedge \cdots \wedge dz_d$ vanishes on $\widetilde{\mathcal{V}}$ wherever $\partial P / \partial f = 0$. Hence, each coefficient $C_{\mathbf{y},0}$ from [PWM23, Theorem 5.3] will vanish, leaving only terms with $\ell \geq 1$.

To argue that the sum in (3.1) is not identically zero, the form of [MW19, (5.2)] implies that for each $\mathbf{y} \in W$, at least one of the coefficients $C_{\mathbf{y},\ell}$ is nonzero. The quantities $e^{-i\mathbf{r}\cdot\mathbf{y}}$ are linearly independent over $\mathbb{C}$ as functions of $\mathbf{r}$, hence nonvanishing of $C_{\mathbf{y},\ell}$ for a single pair $(\mathbf{y}, \ell)$ implies nonvanishing of the double sum. The set contrib varies continuously wherever the hypotheses of the theorem hold, implying uniformity of the estimate (2.7) and finishing the proof of Theorem 3.4. $\qquad \square$

## 4.2  Proof of Theorem 3.6

In proving Theorem 3.6, we also take care of one detail not stated in the theorem, namely how one checks weak minimality of $\mathbf{w}$ under the nonnegativity assumption. Section 4.3 deals with the more difficult algorithm checking minimality in the general case.

Begin by recalling some concepts about polynomial amoebas, e.g., from [GKZ08, Mik04]. Define the coordinatewise log magnitude map Relog by

$$\mathrm{Relog}(\mathbf{z}) := (\log|z_1|, \ldots, \log|z_d|) \,.$$

The **amoeba** of a polynomial function $g : \mathbb{C}^d \to \mathbb{C}$ is the image of the zero set of $g$ under the Relog. The amoeba is a closed set whose complement is divided into finitely many open convex connected components. If $g(\mathbf{0}) \neq 0$ then there is a unique component $G_0$ of the complement of $\mathsf{amoeba}(g)$ and a real number $M$ such that $G_0$ contains $(-\infty, M]^d$. Such a number $M$ can be effectively computed. The torus $T$ in (1.1) can be taken to be the product of circles with radius $e^M$.

Fix a critical point $\mathbf{z}$ in direction $[\mathbf{r}]$ on the variety $\mathcal{V}$ defined by $g$. A more general definition of minimality is that $\mathbf{x} = \mathrm{Relog}\,\mathbf{z}$ lie on the boundary of $\mathsf{amoeba}(g)$, and for an ordinary power series, specifically on $\partial G_0$. When $\mathbf{z} \in \mathcal{V}$, the condition $\mathbf{x} \in \partial G_0$ is equivalent to $\mathcal{D}(\mathbf{z})$ being in the domain of convergence of the power series for $1/g$, where $\mathcal{D}$ is the open polydisk $\{\mathbf{w} : |w_j| < |z_j|, 1 \leq j \leq d\}$. For functions $F$ not necessarily rational but represented by power series absolutely convergent in some domain $\mathcal{D}$ containing the origin (necessarily the union of tori), minimality again generalizes to the condition that $\mathbf{z} \in \partial\mathcal{D}$.

When all coefficients of a power series are known to be nonnegative, checking minimality is particularly easy. This is the case for the power series $F$ in all our examples, but unfortunately this does not help as much as one might think because the minimality testing we need to do is for $g = \mathrm{discr}(P)$ or $g = p_m \cdot \mathrm{discr}(P)$, in neither case leading to a series likely known to have nonnegative coefficients. We therefore also require a condition to ensure minimality upstairs and downstairs are the same, which will be seen below to follow from the second and fourth conditions in Table 1.

14

PROOF OF THEOREM 3.6: The multivariate version of Pringsheim's Theorem (see, e.g., [PWM23, Proposition 6.38]) says that if the power series coefficients of a function $F$ are known to be nonnegative, and if $\mathbf{z} = \exp(\mathbf{x} + i\mathbf{y})$ and $\mathbf{x} \in \partial G_0$, then the positive real point $|\mathbf{z}| := (|z_1|, |z_2|, \ldots, |z_d|)$ is singular for $F$. For algebraic functions, this means $\mathbf{z}$ is a branchpoint or a pole.

Let $\mathbf{z}$ and $\mathbf{w}$ be as in the statement of the theorem. It is shown in [Mel21, Proposition 5.5.4] that weak minimality of $\mathbf{w}$ along with nonnegative coefficients implies minimality of $\mathbf{w}$ (hence $\mathbf{z}$) for $F$. Letting $T'$ denote the torus in $\mathbb{C}_*^d$ through $\pi(\mathbf{z})$, this implies that the lifting of the homotopy $t \cdot T', 0 < t < 1$ does not intersect $\widetilde{\mathbf{br}}$. Under the assumption that $P$ is quadratic in $f$, this is equivalent to the homotopy $t \cdot T', 0 < t < 1$ avoiding $\mathbf{br}$. Because there are only two branches, we know that $\mathbf{p} \in \mathbf{br}$ and $(\mathbf{p}, f) \in \widetilde{\mathcal{V}}$ imply $(\mathbf{p}, f) \in \widetilde{\mathbf{br}}$. From the assumption that $p_m$ is a monomial, we conclude that $\mathbf{pole}$ is empty in $\mathbb{C}_*^d$, hence $\pi(\mathbf{z})$ is minimal for $p_n \cdot \mathrm{discr}(P)$; in fact it is minimal in direction $[\mathbf{r}]$ by Lemma 4.1. Again, because there are only two branches, $\pi^{-1}(\mathbf{z}) \cap \widetilde{\mathcal{V}}$ contains a single point $(\mathbf{z}, f_1)$, and at this point the two solutions with different initial conditions merge. Hence this solution is on the branch determined by either initial condition. □

## 4.3 Verification of minimal points

When $P$ is not quadratic, the correspondence between the branch locus and the vertical tangent locus may not be complete. In this case minimality upstairs and downstairs need not coincide and one would need nonnegativity of coefficients of $1/g$ as well as of $F$, in order to test minimality both upstairs and downstairs. There is no reason to expect $1/g$ to have nonnegative coefficients.

Minimality is effectively testable regardless of any nonnegativity condition, because it is a real semi-algebraic condition. To make full use of the results of this paper requires such an algorithm. We briefly describe one presented by Melczer, referring the reader to [Mel21, Section 7.1.3] for details.

We let $g_\Re$ and $g_\Im$ denote the unique polynomials in the $2d$ real variables $x_1, \ldots x_d$ and $y_1, \ldots y_d$ such that $g(\mathbf{x} + i\mathbf{y}) = g_\Re(\mathbf{x}, \mathbf{y}) + i g_\Im(\mathbf{x}, \mathbf{y})$. Augment the original $d$ complex critical point equations for $g$ ($d$, not $d+1$ as we are dealing with $g$ rather than $P$) to $d+1$ equations via a complex parameter $N$ describing the ratio between $\nabla_{\log} g(\mathbf{p})$ and $\hat{\mathbf{r}}$. These $d+1$ critical point equations expand to $2d+2$ equations in $\mathbf{x}$ and $\mathbf{y}$ and the real and imaginary parts of $N$, involving $g_\Re$, $g_\Im$ and their partial first derivatives in place of $g$ and its derivatives. Thus far these still describe all critical points of $g$. Next one write equation for another real solution on a linearly shrunken torus via the $d+2$ equations $(x_j')^2 + (y_j')^2 = t(x_j^2 + y_j^2)$ for $1 \le j \le d$ and $g_\Re(\mathbf{x}', \mathbf{y}') = g_\Im(\mathbf{x}', \mathbf{y}') = 0$.

This gives $3d+4$ equations in $4d+3$ variables, whose real solutions give pairs of critical points on similar tori. It is shown that minimal critical points for $g$ in direction $\hat{\mathbf{r}}$ correspond precisely to real solutions $(\mathbf{x}, \mathbf{y})$ of the critical point equations for which there are no real solutions $(\mathbf{x}', \mathbf{y}', t)$ to the pair equations with $t \in (0, 1)$. This criterion has alternating quantifiers ranging over infinite sets so is not yet algorithmic. However, introducing a new parameter $\nu$ and $d$ representing the common argument in each coordinate of a hypothetical smaller solution in a pair, and $d$ more equations to force this to be the argument, one obtains a set of $4d+4$ equations in $4d+4$ variables whose real solutions with $t \in (0, 1)$ correspond to refutations of minimality for $(\mathbf{x}, \mathbf{y})$. This results in an algorithm which will either determine some nondegeneracy hypotheses have not been met, or will find all real solutions, eliminate refuted solutions, and correctly produce a set of minimal critical points in direction $\hat{\mathbf{r}}$.

We summarize in a proposition.

**Proposition 4.5** ([Mel21, Section 7.1.3]). *Under mild nondegenearcy assumptions, testing for minimality can be accomplished by searching for real solutions to $4d + 4$ polynomial equations in $4d + 4$ variables, constrained to an interval. Such real semi-algebraic equation solving is effective and is carried out in pseudo-code in [Mel21, Algorithm 3].* □

# 5 Worked examples

Before presenting examples to exhibit the main results, we give a couple of examples showing that algebraic functions are not always determined by integrals through branchpoints.

**Example 5.1.** *In one variable, let $F(x) = (1-x)^{-3/2}$. The defining polynomial is $P(x, f) := (1-x)^3 f^2 - 1 = 0$, with $\mathrm{discr}(P) = 4(1 - x)^3$. The unique branchpoint $x = 1$ is also a pole: $\mathbf{br} = \mathbf{pole} = \{1\}$, so integration through this point is not possible. Not coincidentally, the hypothesis that $\mathrm{discr}(P)$ be squarefree is also violated.*

**Example 5.2.** *The generating function*

$$\beta(x, y, z) = \frac{1}{\sqrt{(1 - x - y)^2 - 4xy - z}}$$

*generates certain hypergeometric sums arising when counting solutions to Ulam's problem [HS22]. The discriminant locus is where the quantity under the radical vanishes. The locus of vanishing is the parabola inscribed in the positive $x$-$y$-quadrant and tangent at $(0, 1)$ and $(1, 0)$; see Figure 2. The pole locus is where the quantity under the radical is equal to $z$. For fixed $z \in (0, 1)$ this vanishes on an arc in the positive quadrant hitting the axes at $1 - \sqrt{z}$. The minimal points of $\mathbf{pole} \cup \mathbf{br}$ are the ones on this arc, where $\beta$ has a pole, not a branchpoint. The coefficient asymptotics there are governed by the usual ACSV smooth point formula given, for example, in [PW13, Theorem 9.2.7].*
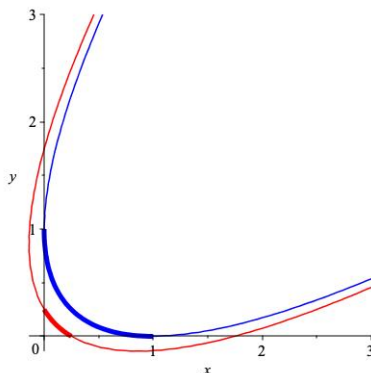


Figure 2: pole variety (red) and branching locus (blue)

The remainder of the section gives examples of the application of Theorems 3.4 and 3.6 and Corollary 3.5.

## 5.1 Toy example: Catalan GF

In this example $d = 1$ so the generating function $F$ is a univariate algebraic function, for which well known methods such as the transfer theorems of Flajolet and Odlyzko [FO90] could be applied (see also far earlier works). This example illustrates our methods in the simplest case.

Let $F(z) = (1 - \sqrt{1 - 4z})/(2z) = \sum_{n=0}^{\infty} a_n z^n$ be the generating function for the Catalan numbers $a_n := \binom{2n}{n}/(2n+1)$. A minimal polynomial representing $F$ via $P(F, z) = 0$ and $F(0) = 1$ is

$$P(F, z) := zf^2 - f + 1.$$

The discriminant of $P$ is $1 - 4z$ and the pole polynomial is $z$. A graph of $F$ in the $\mathbb{R} \times \mathbb{R}$ subspace of $\mathbb{C}^2$ is shown on the left of Figure 3.
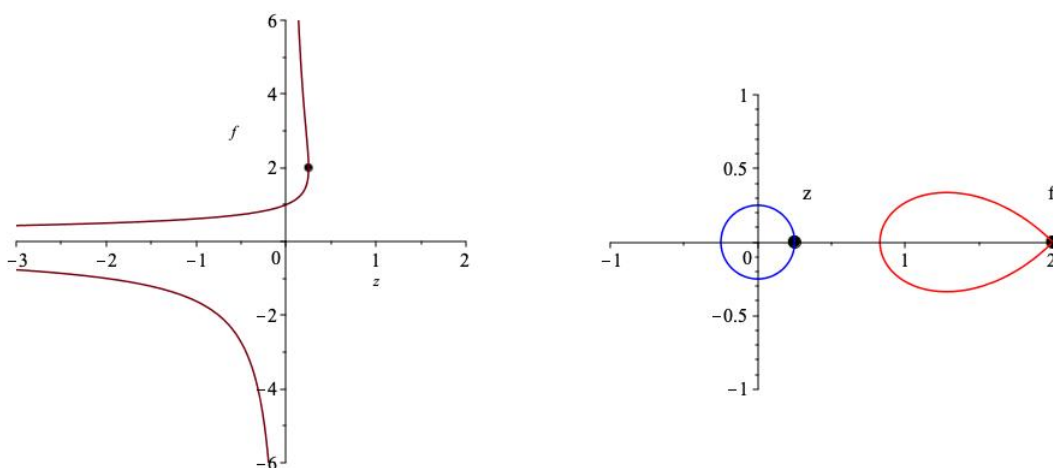


Figure 3: Left: $\mathbb{R} \times \mathbb{R}$ graph of the Catalan GF; Right: coordinates in $\mathbb{C}$ of the lifted torus

Here, $m = 2$ but because the origin is a zero of $p_m$, there is only one function $F$ with a finite value at the origin. This branch has $F(0) = 1$, while the other possible branch, $(1 + \sqrt{1 - 4z})/(2z)$, has a pole at zero. The discriminant of $P$ with respect to $f$ is $1 - 4z$, whose amoeba is the singleton set $\{\log(1/4)\}$. Thus a circle of radius $\varepsilon$ about the origin, call it $T$, may be expanded without hitting without hitting **br** or **pole** until it has radius $1/4$, which circle we call $T'$. Lifting the homotopy from $T$ to $T'$ into the algebraic curve $P(z, f) = 0$ yields a homotopy

$$Y_t(\theta) := (te^{i\theta}, F(te^{i\theta}))$$

between circles $\tilde{T}$ and $\tilde{T}'$ within $\widetilde{\mathcal{V}}$.

We are now in a position to apply Theorem 3.4. This toy example is intended to explain the working parts of the theorem, hence instead of actually applying the theorem we will follow the proof of the theorem to derive the asymptotic expansion from Proposition 2.6 and Corollary 2.7. We begin by determining the oriented stationary phase integral defined by the contour $\tilde{T}'$.

For $z$ positive and sufficiently small, the branch of $f$ that blows up at $z = 0$ takes positive real values, hence the branch defining $F$ will be the lower of two branches coalescing at 2 when $z = 1/4$. Because $F(z)$ is the smaller real root of $P(z, f) = 0$ for small positive real $z$, Lemma A.1 implies that $\tilde{T}'$ is positively oriented

17

with respect to $d(if)$. To see the significance of this, look on the right of Figure 3, where the $z$ and $f$ coordinates of the circle $\tilde{T}' \subseteq \tilde{\mathcal{V}}$ are shown. The projection of $\tilde{T}'$ to the $f$-coordinate is nondifferentiable at $z = 1/4$ because the square root in $f = (1 \pm \sqrt{1-4z})/(2z)$ is the principal root (the one lying in the right half plane); mapping the blue circle by $1 - 4z$ gives a circle tangent to the imaginary axis and lying in the closed right half-plane, whence taking the principal square root produces a discontinuity in the argument resulting in the nondifferentiability in the figure.

The phase $\phi(z, f) = \log|z|$ on the circle $\tilde{T}'$ is constant. We can further deform $\tilde{T}'$ so that the $f$-coordinate adheres locally to the line $\Re(f) = 2$, and so that the minimum of $h_{\hat{\mathbf{r}}}$ is achieved strictly at $(1/4, 2)$. This illustrates that the imaginary direction is always a direction along which the height, in this case $\log|z|$, will have a local minimum.

Contours with phase minimized at $(1/4, 2)$ are of course not unique. We could deform the $f$-coordinate to a circle $f = 2e^{i\theta}$ and the $z$-coordinate to $z = (f-1)/f^2$, for example, however in this case it is simplest to choose the chain $\Gamma$ obtained from deforming a small arc on the right of the circle to lie on the segment $f = 2 + it, -\varepsilon \le t \le \varepsilon$. Along with $z = (f-1)/f^2$, this defines a parametrization of a curve, along which the derivatives in Corollary 2.7 are computed with minimal effort. Reparametrizing everything in terms of $f$, we get $dz = (2-f)df/f^3$ and $(f/z)dz = [(2-f)/(f-1)]df$ and (4.1) becomes

$$
\begin{aligned}
a_n &= \frac{1}{2\pi i} \int_\Gamma z^{-n} f \frac{dz}{z} \\
&= \frac{1}{2\pi i} \int_\Gamma \frac{2-f}{f-1} e^{-n\phi(f)} \, df
\end{aligned}
$$

where $\phi(f) = \log z = \log((f-1)/f^2)$ has power series expansion around $f = 2$ given by

$$
\phi(f) \approx -\log 4 - \frac{1}{4}(f-2)^2 + \frac{1}{4}(f-2)^3 + O(f-2)^4.
$$

To fit into the set-up of Proposition 2.6, we need to recenter the critical point $z = 1/4, f = 2$ at the origin. So by a shift of the coordinate, we get

$$
\begin{aligned}
a_n &= \frac{1}{2\pi i} \int_{\Gamma'} \frac{-f}{f+1} e^{-n\phi(f+2)} \, df && (5.1) \\
&= \frac{1}{2\pi i} e^{n\log(4)} \int_{\Gamma'} \frac{-f}{f+1} e^{-n\Phi(f)} \, df && (5.2) \quad \text{\{eq:catalan\}}
\end{aligned}
$$

where $\Phi(f) = \phi(f+2) + \log(4)$ has power series expansion at the origin given by

$$
\Phi(f) \approx -\frac{1}{4}f^2 + \frac{1}{4}f^3 + O(f)^4.
$$

Now we pull back the integral $\int_{\Gamma'} \frac{-f}{f+1} e^{-n\Phi(f)} \, df$ to $\mathbb{R}$. We obtain

$$
\int_{\Gamma'} \frac{-f}{f+1} e^{-n\Phi(f)} \, df \approx \int_\epsilon^\epsilon \frac{-it}{1+it} e^{-n\Phi(it)} \, i\, dt
$$

where $\Phi(it) = \log(\frac{1+it}{(2+it)^2}) + \log(4)$. Here, we use $\approx$ and $\epsilon$ because the portion of $\Gamma$ away from the critical point contributes exponentially small value to the integral. We can recognize the amplitude function $A(t) = \frac{-it}{1+it}$ and the phase function $\Phi(it)$.

18

In Corollary 2.7, $V = 1/2$, $A'(0) = -i$, $A''(0) = -2$, $\Phi'''(0) = -\frac{3i}{2}$. By Corollary 3.5, the choice of $V^{1/2}$ should be the positive one. By (2.10), the leading constant $C_1$ for the integral is $\sqrt{2}$. Therefore, Proposition 2.6 gives

$$
\begin{aligned}
a_n &\approx \frac{1}{2\pi} e^{n \log(4)} \int_\epsilon^\epsilon \frac{-it}{1+it} e^{-n\Phi(it)} \, dt \\
&\approx \frac{1}{2\pi} 4^n \left( \frac{2\pi}{n} \right)^{1/2} \left( \sqrt{2} \, n^{-1} \right) \\
&= \frac{4^n}{\sqrt{\pi}} n^{-3/2}
\end{aligned}
$$

To sum up, we have derived asymnptotics for Catalan numbers by constructing an explicit lifting of the univariate Cauchy integral to an integral within the algebraic curve $zf^2 - f + 1 = 0$ that witness the Catalan generating function as algebraic.

Theorem 3.4 automates most of this procedure. Having checked that $(1/4, 2)$ is a minimal point, we apply the theorem to (4.1) using Proposition 2.6 and its Corollary 2.7 for the computation. Plugging in $A(f) = \frac{-f}{f+1}$ and $\Phi(f) = \log((f+1)/(f+2)^2) + \log(4)$ immediately yields $a_{\mathbf{r}} = \pm 4^n n^{-3/2}/\sqrt{\pi}$. The sign (should it be in doubt) is determined by Lemma A.1 to be positive because the generating function $F(z) = (1 - \sqrt{1-4z})/(2z)$ is the lower of two branches in a neighbrhood of the origin.

## 5.2   Assembly trees

{ss:assembly}

In this example $d = 2$ and $F$ is the generating function for the number of assembly trees of the complete bipartite graph, counted by the sizes of the left and right vertex sets [BV13]. This generating function is given by

$$
F(x, y) = 1 - \sqrt{(1-x)^2 + (1-y)^2 - 1}
$$

satisfying an obvious polynomial equation

$$
P(F, x, y) := F^2 - 2F - x^2 - y^2 + 2x + 2y = 0 \, .
$$

The discriminant is given by

$$
\operatorname{discr}(P) = (x-1)^2 + (y-1)^2 - 1 \, .
$$

The branch locus $\mathbf{br}$ where this vanishes is the circle of radius 1 centered at $(1, 1)$. The defining variety $\widetilde{\mathcal{V}}$ is the hyperboloid $\{P = 0\}$ and the vertical tangent locus in $\widetilde{\mathcal{V}}$, which projects to $\widetilde{\mathbf{br}}$, is the intersection of the hyperboloid with the horizontal plane $\{f = 1\}$. The branch locus is depicted on the left of Figure 4, with $\widetilde{\mathcal{V}}$ and $\widetilde{\mathbf{br}}$ shown on the right.

Given a direction $\hat{\mathbf{r}} = (\hat{r}, 1 - \hat{r})$, there is a corresponding minimal point $(x, y)$ on the branch locus, specifically on the quarter-circle arc joining $(1, 0)$ to $(0, 1)$. This point is given by

$$
x(\hat{\mathbf{r}}) = \frac{1}{2} + \hat{r} - \frac{\sqrt{1 + 4\hat{r} - 4\hat{r}^2}}{2} \qquad ; \qquad y(\hat{\mathbf{r}}) = \frac{3}{2} - \hat{r} - \frac{\sqrt{1 + 4\hat{r} - 4\hat{r}^2}}{2} \, .
$$

The point $(x(\hat{\mathbf{r}}), y(\hat{\mathbf{r}}))$ lifts to a unique point $(x(\hat{\mathbf{r}}), y(\hat{\mathbf{r}}), 1)$ in the hyperboloid where the two solutions of $P(x, y, f)$ coincide. It follows from (3.1) that, uniformly when $r/s$ and $s/r$ are bounded,

$$
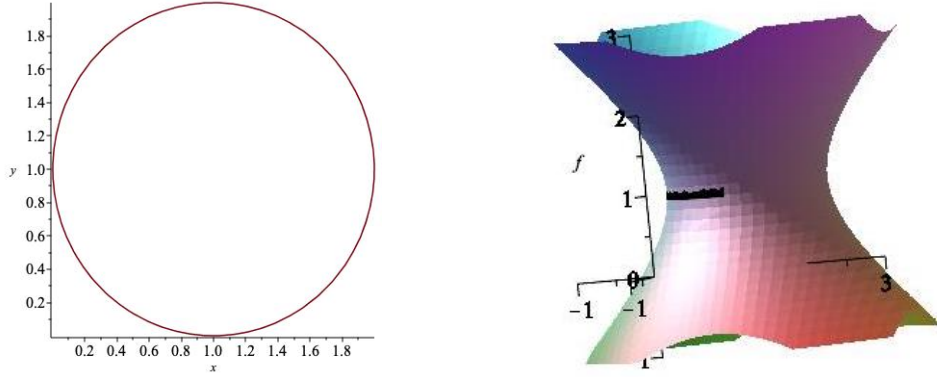a_{r,s} \sim C(r+s)^{-2} e^{-r \log x(\hat{\mathbf{r}}) - s \log y(\hat{\mathbf{r}})}
$$

19

Figure 4: Left: the branch locus $\mathbf{br}$; Right: $\widetilde{\mathcal{V}}$, with $\widetilde{\mathbf{br}}$ shown in black

where $\hat{r} := r/N$, $N := r + s$, and $C$ is given by computing the stationary phase integral (4.1) on a contour passing through $\mathbf{p} := (x(\hat{\mathbf{r}}), y(\hat{\mathbf{r}}), 1)$.

Parametrizing $\widetilde{\mathcal{V}}$ near $\mathbf{p}$ by $x$ and $f$ (we could have chosen $f$ and any $ax + by$ other than the one orthogonal to $\widetilde{\mathcal{V}}$) we rewrite $f\,dx\,dy$ as $f J\,dx\,df$ where

$$J = \frac{\partial y}{\partial f} = -\frac{\partial P/\partial f}{\partial P/\partial y}.$$

The exponential factor $e^{-r\log x(\hat{\mathbf{r}}) - s\log y(\hat{\mathbf{r}})} = \left(x(\hat{\mathbf{r}})^{-\hat{r}} y(\hat{\mathbf{r}})^{\hat{r}-1}\right)^N$ comes from the stationary phase integral $\int A(x, f) e^{-N\phi(x,f)}dx\,df$ by Corollary 3.5 where $\phi(x, f) = \hat{r}\log x + (1 - \hat{r})\log(y(x, f))$ and

$$A(x, f) := \frac{f \cdot J}{x \cdot y(x, f)} = \frac{f(1 - f)}{x\,y(x, f)(1 - y(x, f))};$$

here, $y = 1 - \sqrt{1 + (1 - f)^2 - (1 - x)^2}$, taking the principal root which is well defined near $(x(\hat{\mathbf{r}}), y(\hat{\mathbf{r}}), 1)$.

One can use either Proposition 2.6 or Corollary 2.7 to calculate the constant $C_1$. In practice, we use Corollary 2.7. This avoids computing $\mathbb{H}^2(A \cdot f^2)$, which is not only messy but wasteful, computing out to four partial derivatives when Corollary 2.7 shows that only third partial derivatives are required. Also, many of the partial derivatives in (2.11) arise only in products with other partial derivatives, meaning that the vanishing of some partial derivatives allow us to avoid the computation of many more. In this case, for example, vanishing partial derivatives lead to the following simple expression for $C_1$, all partial derivatives being evaluated at $x = x(\hat{\mathbf{r}}), f = 1$.

$$C_1 = \frac{A_{ff}\phi_{xx}}{2}\,\left(\phi_{xx}\phi_{ff}\right)^{-\frac{3}{2}} \tag{5.3}$$

Having reduced the computation to the evaluation of partial derivatives of algebraic expressions, we illustrate how computer algebra systems handling polynomial computations via Gröbner bases can be harnessed to differentiate algebraic functions. Note that built-in differentiation operators in computer algebra systems such as Sage and Maple do not handle radicals well. Compare (5.4), for example, with the expression taking up a full line in line 45 of the online worksheet attached to [GMRW22].

The idea is that if the arguments to a function $B(x_1, \ldots, x_k)$ are algebraic expressions in other variables $y_{i,j}, 1 \le i \le k, 1 \le j \le m_i$, then implicit differentiation can be used to compute derivatives of B as rational

functions of all the variables involved. One can then clear denominators and eliminate the $y$ variables to obtain an algebraic representation for the derivatives of $B$. To illustrate: in the present example, we need to compute $A_{ff}(x, f)$; originally $A$ was represented as a rational function of $x$ and $y$; reparametrizing by $x$ and $f$ requires substituting an algebraic expression in $x$ and $f$ for $y$, via $P(x, y, f) = 0$. We then use the identity

$$\frac{\partial A(x, y(x, f), f)}{\partial f} = \frac{\partial A(x, y, f)}{\partial f} + \frac{\partial A(x, y, f)}{\partial y}\frac{\partial y(x, f)}{\partial f}.$$

Implicitly differentiating $P$, this becomes

$$\frac{\partial A(x, y(x, f), f)}{\partial f} = \frac{\partial A(x, y, f)}{\partial f} - \frac{\partial A(x, y, f)}{\partial y}\frac{\partial P/\partial f}{\partial P/\partial y}..$$

Computing terms in (5.3) in this way represents $A_{ff}, \phi_{xx}$ and $\phi_{ff}$ as rational functions of $x, f$ and $y$. A Gröbner basis computation using $P$ to eliminate $y$, and using the critical point equations to eliminate $f$ and $x$, one obtains a polynomial satisfied by $r$ and $A_{ff}$, another satisfied by $r$ and $\phi_{xx}$ and a third satisfied by $r$ and $\phi_{ff}$. These polynomials are quadratic, leading to the following solutions by radicals:

$$A_{ff}(x(\hat{\mathbf{r}}), 1) = \frac{(-\hat{r} - 1)\sqrt{-4\hat{r}^2 + 4\hat{r} + 1} + 2\hat{r}^2 - 3\hat{r} - 1}{4\hat{r}^3(\hat{r} - 1)^2}$$

$$\phi_{ff}(x(\hat{\mathbf{r}}), 1) = \frac{1 + \sqrt{-4\hat{r}^2 + 4\hat{r} + 1}}{4\hat{r}(\hat{r} - 1)}$$

$$\phi_{xx}(x(\hat{\mathbf{r}}), 1) = \frac{(4\hat{r}^2 - 2\hat{r} - 1)\sqrt{-4\hat{r}^2 + 4\hat{r} + 1} + 4\hat{r}^2 - 4\hat{r} - 1}{16\hat{r}^3(\hat{r} - 1)^2}$$

These three polynomials, along with the polynomial relation between $A_{ff}, \phi_{xx}, \phi_{ff}$ and $C_1$ obtained from squaring (5.3) and clearing denominators gives an elimination polynomial $\left(\hat{r}^2 - \hat{r} - \frac{1}{4}\right)C_1^4 + 4$ satisfied by $C_1$ and $\hat{r}$, yielding the radical expression

$$C_1 = -\frac{2}{(1 + 4\hat{r} - 4\hat{r}^2)^{1/4}}. \tag{5.4} \quad \texttt{\{eq:simpleC1\}}$$

Putting this all together, using $\phi(x(\hat{\mathbf{r}}), 1) = \hat{r}\log(x(\hat{r})) + (1 - \hat{r})\log(y(\hat{r}))$ along with equation (4.1) and Proposition 2.6, yields

$$a_{\mathbf{r}} \approx \frac{-C_1}{2\pi}\frac{\left(x(\hat{\mathbf{r}})^{-\hat{r}}y(\hat{\mathbf{r}})^{\hat{r}-1}\right)^N}{N^2}.$$

Expanding,

$$a_{\mathbf{r}} \approx \left(\frac{1}{2} + \hat{r} - \frac{\sqrt{1 + 4\hat{r} - 4\hat{r}^2}}{2}\right)^{-N\hat{r}}\left(\frac{3}{2} - \hat{r} - \frac{\sqrt{1 + 4\hat{r} - 4\hat{r}^2}}{2}\right)^{N(\hat{r}-1)}\left(\frac{1}{(1 + 4\hat{r} - 4\hat{r}^2)^{1/4}\pi}N^{-2}\right)$$

where $N = |\mathbf{r}|$ and $\hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|}$. For example, when $\hat{r} = 1/2$ and $N$ is even, $a_{N/2, N/2} \approx 3.4142^N(0.2677\,N^{-2})$, agreeing with the value given by the considerably more complicated expression in [GMRW22, online attachment].

## 5.3  Bi-colored Motzkin paths

A bicolored Motzkin path on the $x$-$y$ plane starts at the origin and ends at the $x$-axis, never goes below the x-axis and takes steps $U = (1,1)$, $D = (1,-1)$, and two colored horizontal steps $H_1 = (1,0)$ and $H_2 = (1,0)$. Let $\mathcal{M}^2$ be the set of bicolored Motzkin paths. Define $u(M), d(M), h_1(M), h_2(M)$ to be the number of $U, D, H_1, H_2$ steps in the bicolored Motzkin path $M \in \mathcal{M}^2$ respectively. The generating function $F(x,y) = \sum_{M \in \mathcal{M}^2} x^{d(M)+h_1(M)} y^{u(M)+h_2(M)}$ is counting the number of paths by the total number of $D$ and $H_1$ steps and the total number of $U$ and $H_2$ steps. In particular, $[x^i y^j]F(x,y)$ is the number of such paths with $i$ steps in $D$ and $H_1$ and $j$ steps in $U$ and $H_2$. [Eli21, Lemma 2.1] shows that

$$F(x,y) = \frac{1 - x - y - \sqrt{(1-x-y)^2 - 4xy}}{2xy}.$$

Let $\hat{\mathbf{r}} = (\hat{r}, 1-\hat{r})$ and let $\mathbf{r} = (r,s) = N\hat{\mathbf{r}}$. We calculate the asymptotic formula for $a_{\mathbf{r}} = [x^{\hat{r}N} y^{(1-\hat{r})N}]F(x,y)$. The minimal polynomial $P(x,y,f)$ satisfying $P(x,y,F(x,y)) = 0$ is

$$P(x,y,f) = xyf^2 + (x+y-1)f + 1.$$

Notice that $P$ satisfies all four assumptions in Table 1.

The discriminant is given by

$$\mathrm{discr}(P) = (1-x-y)^2 - 4xy.$$

Given $\hat{\mathbf{r}} = (\hat{r}, 1-\hat{r})$ with $0 < \hat{r} < 1$, there is one minimal critical point on the branch locus, given by

$$x(\hat{\mathbf{r}}) = \hat{r}^2, \quad y(\hat{\mathbf{r}}) = \hat{r}^2 - 2\hat{r} + 1.$$

The point $(x(\hat{\mathbf{r}}), y(\hat{\mathbf{r}}))$ lifts to a unique point $\mathbf{p} = (x(\hat{\mathbf{r}}), y(\hat{\mathbf{r}}), f(\hat{\mathbf{r}}))$ where $f(\hat{\mathbf{r}}) = -\dfrac{\hat{r}^2 - \hat{r} + 1}{(\hat{r}-1)\hat{r}}$. The branch defining $F$ is the lower of two branches, the other being at $+\infty$. Hence, the rank of $f_0 = F(0,0) = 1$ is 2. Therefore, when we apply Corollary 3.5, we don't need to flip the sign.

Next, we parametrize near $\mathbf{p}$ by $x$ and $f$ coordinates. The Jacobian is $J = \partial y / \partial f$ is $-P_f/P_y$, the amplitude function is $A(x,f) = fJ/(x \cdot y(x,f))$, and the phase function is $\phi(x,f) = \hat{r}\log(x) + (1-\hat{r})\log(y(x,f))$. By Proposition 2.6, equation (4.1) becomes

$$
\begin{aligned}
a_{\mathbf{r}} &\approx \left(\frac{1}{2\pi i}\right)^2 e^{-N\phi(x(\hat{\mathbf{r}}), f(\hat{\mathbf{r}}))} \left(\frac{2\pi}{N}\right) C_1 N^{-1} \\
&= \left(\frac{1}{2\pi i}\right)^2 (x(\hat{\mathbf{r}})^{-N\hat{r}} y(\hat{\mathbf{r}})^{-N(1-\hat{r})}) \left(\frac{2\pi}{N}\right) C_1 N^{-1} \\
&= -\frac{1}{2\pi} \left(\hat{r}^{-2N\hat{r}} (\hat{r}-1)^{2N(\hat{r}-1)}\right) C_1 N^{-2},
\end{aligned}
$$

where the constant $C_1$ is computed by Corollary 2.7. None of partial derivatives of $A$ and $\phi$ involved in (2.11) of Corollary 2.7 vanishes at the critical point. However, when pieced together, they yield a simple form for the constant $C_1$:

$$C_1 = -\frac{1}{2(\det H)^{1/2}} \frac{2}{\hat{r}^2(\hat{r}-1)}.$$

Furthermore, $\det H = (1-\hat{r})^2$, with Corollary 3.5 specifying that the choice $(\det H)^{1/2}$ is to be interpreted as $-\sqrt{\det H}$, in other words, the negative real root. Therefore, $C_1 = -1/(\hat{r}^2(\hat{r}-1)^2)$ and

$$a_{\mathbf{r}} \approx \left(\frac{1}{\hat{r}^{2\hat{r}} \, (1-\hat{r})^{2(1-\hat{r})}}\right)^N \frac{1}{2\pi \hat{r}^2 (\hat{r}-1)^2} N^{-2}.$$

22

## 5.4   0-2-5 Trees

The usual definition of a binary tree is a rooted tree in which each vertex has either zero or two children. The number of binary trees with $n$ nodes is the $n$th Catalan number, due to the recursion satisfied by binary trees, as follows. If we allow the empty tree, a binary tree is either empty, or a root with a left and right subtree. Thus, the generating function $f(z) := \sum_{n=0}^{\infty} a_n z^n$ for the numbers $a_n$ of $n$-vertex binary trees is equal to $1 + zf(z)^2$.

One can generalize this to allow the number of children to be either zero or a member of a given subset $E$ of the positive integers. Such trees can be counted by the number of nodes having $m$ children, for each $m \in E$. When $E$ is a finite set, this leads to an algebraic generating function in finitely many variables. The numbers $n_j$ of vertices having $j$ children in such a tree, if non-empty, satisfy the linear relation:

$$N := n_0 + \sum_{j \in E} n_j = 1 + \sum_{j \in E} j n_j.$$

For this reason, the number of independent statistics by which these trees can be counted is $|E|$ rather than $|E| + 1$; for example, one might keep track of $N$ and all but one of the $n_j$.

This subsection analyzes one example, $E = \{2, 5\}$, in other words, trees where every vertex has zero, two or five children. Let $a_{ij}$ count 0-2-5 trees $T$ with $N(T) = j$ and $n_5(T) = i$. Let $F(y, z) = \sum_{i,j \geq 0} a_{ij} y^i z^j$. The relation $F(y, z) = 1 + z[((F(y, z) - 1)^2 + 1) + y(F(y, z) - 1)^5]$ follows from the recursive description of a 0-2-5 tree as either being empty or consisting of a root and either zero or two trees (counted by 1 or $(F - 1)^2$, respectively), or five subtrees (counted by $(F - 1)^5$ with an extra factor of $y$ to keep track of the addition of a 5-child vertex). The total number $N(T)$ of a nonempty such tree $T$ is equal to $1 + 2n_2(T) + 5n_5(T)$.

When counting by $N$ and $n_5$, there is a periodicity because $N - 1 - 5n_5 = 2n_2$ implies $N + n_5 \equiv 1 \mod 2$. Therefore, the generating function has nonconstant terms only of odd total degree[6]. In classical generating function singularity analyses, periodicity is reflected by summing asymptotic series around more than one critical point; the coordinates of the critical points differ by factors that are roots of unity; see, e.g., [Sta97] for a discussion of the univariate case. The same holds for the present analysis. Because $F - 1$ is an odd function, critical points come in pairs $(x, y)$ and $(-x, -y)$, whose asymptotics series exactly cancel in even degrees (as we know they must, to produce zero) and are equal in odd degrees.

Let $P(y, z, f) = 1 - f + z[((f - 1)^2 - 1) + y(f - 1)^5]$. Then $P(y, z, F) = 0$ and $F(0, 0) = 1$ uniquely specifies $F$ as an algebraic function, analytic in a neighborhood of the origin in $\mathbb{C}^2$. We chose the class of 0-2-5 trees to analyze rather than, say, 0-2-3 trees, in order to show that nothing changes when $F$ cannot be expressed via radicals. The term with the highest degree in $f$ is the $f^5$ term, whose coefficient is $xy$. Therefore, the pole variety **pole** is the union of two coordinate axes, and the expanding torus will not run into it. Using computer algebra, we verify that $P$ defines a smooth variety and compute the branch locus **br** defined by the discriminant

$$\mathrm{discr}(P) = 3125y^2 z^5 - 2250yz^4 + 108z^5 + 1600yz^2 - 27z^3 - 256y.$$

As expected, this is an odd function, meaning that $\mathrm{discr}(P)(-y, -z) = -\mathrm{discr}(P)(y, z)$, and leading to an antipodal symmetry among roots $(y, z)$ of the discriminant. However, the polynomial $P$ has degree 5 in $f$,

---

[6]Indeed, this can be detected directly from the annihilating polynomial $P$. In this case, for example, letting $F = G - 1$, the defining polynomial is $G = z(1 + G^2 + yG^5)$; the support (the exponents of monomials) is $\{(0, 0, 1), (1, 0, 0), (1, 0, 2), (1, 1, 5)\}$, which is contained in the odd sublattice of $\mathbb{Z}^3$.

violating the last assumption in Table 1 and requiring extra care. To carry out the branch determination in Conclusion 1 of Theorem 3.4 we determine the rank of $F(0,0) = 1$ within the ordering of real roots of $P(0,0,\cdot)$. A little extra work is needed in this case because the coefficients of $P(0,0,\cdot)$ vanish beyond degree 1, so four of the five roots are at infinity. Computing Puiseux expansions, we see that the roots of $P(\varepsilon, \varepsilon, \cdot)$ occur at roughly $1 + \varepsilon, \pm\varepsilon^{-1/2}$ and $c \pm i\varepsilon^{-1/2}$. We see that three are real, and that $F(0)$ is the limit of the middle one of the three.

Critical points in the direction $n_5/N = \hat{r}/(1 - \hat{r})$ correspond to the equation $Q := \hat{r}z\,P_z - (1 - \hat{r})y\,P_y = 0$. Computing a Gröbner basis for the ideal generated $Q, P$ and $\partial P/\partial f$ gives the generators $[(4\,\hat{r}^2 - 8\,\hat{r} + 4)\,z^2 + 12\,\hat{r}^2 + 4\,\hat{r} - 1, (216\,\hat{r}^3 - 108\,\hat{r}^2 + 18\,\hat{r} - 1)\,y - (8\,\hat{r}^3 - 4\,\hat{r}^2 - 4\,\hat{r})\,z, (2\,\hat{r} - 2)\,z + (2\,\hat{r} + 1)\,f - 2\,\hat{r} - 1]$, in other words, precisely two points $(y_0, z_0, f_0)$ and $(-y_0, -z_0, 2 - f_0)$. Projected to the $y$-$z$ plane, both of these lie on the same centered torus. We verify, using the techniques described in Section 4.3, that the positive point is a minimal point for $\operatorname{discr}(P)$ in direction $n_5/N = \hat{r}/(1 - \hat{r})$, hence both points are. Computer algebra output of this verification is omitted.

To complete the homotopy continuation, we check that the critical point is on the same branch of $F$ as is $(0, 0, 1)$. It suffices to check this for any single $\hat{r}$ in the feasible interval $(0, 1/6)$, as long as we also check that roots do not coalesce further for any value of $\hat{r}$ in this interval, which follows from checking that along the subset of the curve in the $(y, z)$-plane defined by $\operatorname{discr}(P)$ parametrized by $0 < r < 1/6$, the polynomial $F(x(r), y(r), \cdot)$ always has precisely one doubled root, never more. Computer algebra output is again omitted. Now we set $\hat{r} = 1/11$, obtaining the Gröbner basis $[400\,z^2 - 65, 125y + 520\,z, -13 - 20\,z + 13\,f]$. At the positive real point $(y_0, z_0)$ on this curve, the defining polynomial factors into

$$\left(26\,f^3 + \left(4\sqrt{65} - 78\right)f^2 + \left(108 - 8\sqrt{65}\right)f + 9\sqrt{65} - 56\right)\left(-13\,f + 13 + \sqrt{65}\right)^2$$

Then $f_0$ is the root of the second polynomial, roughly 1.62. This root is doubled and greater than the third real root, which is roughly $-0.28$, coming from the first polynomial. Therefore the branch of $F$ containing the initial condition $(0, 0, 1)$ is the lower of two branches passing through $(y_0, z_0, f_0)$, and lifting a homotopy from $T$ to the torus $T'$ through $(y_0, z_0, f_0)$ yields a homotopy of $\tilde{T}$ to $\tilde{T}'$, a torus in the pre-image $\pi^{-1}T'$ that passes through $(y_0, z_0, f_0)$ with positive orientation and is supported at height lower that $h_{1/11}(y, z)$ elsewhere.

The upshot of this is that we have verified that the coefficients $a_{\mathbf{r}}$ of the generating function $F$ for 0-2-5-trees counted by total nodes and outdegree-5 nodes can be estimated by the integral

$$\left(\frac{1}{2\pi i}\right)^2 \int_{\tilde{T}'} \exp(-N\phi)f\,dz\,dy \tag{5.5}$$

over a positively oriented $\tilde{T}' \subseteq \tilde{\mathcal{V}}$ that passes through the points $(y_0, z_0, f_0)$ and $(-y_0, -z_0, 2 - f_0)$, at which it is stationary for the phase function $\phi = \hat{r}\log y + (1 - \hat{r})\log z$. We may compute the result for the positive point only and double it to estimate all $a_{rs}$ with $r + s$ odd, the even coefficients being zero.

Given a direction $\hat{\mathbf{r}} = (\hat{r}, 1 - \hat{r})$, we calculate the asymptotics of $a_{\mathbf{r}}$ where $\mathbf{r} = N\hat{\mathbf{r}}$, $N\hat{r} \in \mathbb{N}$, and $N(1 - \hat{r}) \in \mathbb{N}$. To make $a_{\mathbf{r}}$ have combinatorical meaning, we restrict $0 < \hat{r} < 1/6$. There are two antipodal minimal critical points on the branch locus defined by the discriminant, the positive one of which is

$$y_0(\hat{\mathbf{r}}) = -\frac{2(2\hat{r} + 1)\hat{r}\sqrt{-12\hat{r}^2 - 4\hat{r} + 1}}{(6\hat{r} - 1)^3} \qquad z_0(\hat{\mathbf{r}}) = -\frac{\sqrt{-12\hat{r}^2 - 4\hat{r} + 1}}{2(\hat{r} - 1)}.$$

24

This lifts uniquely to the vertical tangent locus at $(y_0, z_0, f_0)$; we will not need an explicit expression for $f_0$; moreover, we omit the argument $\hat{\mathbf{r}}$ unless comparing expressions for different directions. The exponential growth rate is $\left(y_0^{-\hat{r}} z_0^{\hat{r}-1}\right)^N$.

By Proposition 2.6, the next thing left to calculate in the integral (5.5) is the constant term $C_1$. We show the $C_1$ for the integral (5.5) at the critical point $(y_0, z_0, f_0)$. That of the integral at the other critical point is the same. We parametrize near $(y_0, z_0, f_0)$ using the $z - f$ coordinates. The Jacobian $J := \partial y / \partial f$ at $(y_0, z_0, f_0)$ is $-P_f/P_y$ evaluated at the point. The amplitude function $A$ is $fJ/(yz)$ and the phase function $\phi$ is $\hat{r}\log(y) + (1 - \hat{r})\log(z)$. Using the techniques of implicit differentiation introduced in Section 5.2, we can calculate every partial derivatives of $A$ and $\phi$ needed in Corollary 2.7. Unlike Section 5.2, none of these partial derivatives vanishes. The good news is that the calculation shows that these partial derivatives at the critical point do not depend on the critical point they are evaluated at. For example, at the critical point, $\phi_{zf} = -8(\hat{r} - 1)^2/(6\hat{r} - 1)$, which doesn't involve any $y, z$ or $f$ in which we need to plug $y_0, z_0$ and $f_0$. All partial derivatives in (2.11) for these two critical points are the same and so the constants $C_1$ are the same. In particular,

$$C_1 = \frac{2(1 - \hat{r})}{\hat{r}\sqrt{1 - 6\hat{r}}\sqrt{1 + 2\hat{r}}(\det H)^{1/2}}$$

where $\det H = \dfrac{4(1 - \hat{r})^3(1 + 2\hat{r})}{(1 - 6\hat{r})\hat{r}}$. The square root on $\det H$ is chosen to be $-\sqrt{\det H}$ by Corollary 3.5. Therefore,

$$C_1 = -\frac{1}{\sqrt{\hat{r}}\sqrt{1 - \hat{r}}(1 + 2\hat{r})}.$$

Combining everything together, by Proposition 2.6, the integral 5.5 at the critical point $(y_0, z_0, f_0)$ is

$$
\begin{aligned}
\left(\frac{1}{2\pi i}\right)^2 \int_{\tilde{T}'} \exp(-N\phi) f \, dz \, dy &\approx \left(\frac{1}{2\pi i}\right)^2 \left(y_0^{-\hat{r}} z_0^{\hat{r}-1}\right)^N \frac{2\pi}{N} C_1 N^{-1} \\
&= \frac{\left(y_0^{-\hat{r}} z_0^{\hat{r}-1}\right)^N}{2\pi\sqrt{\hat{r}}\sqrt{1 - \hat{r}}(1 + 2\hat{r})} N^{-2}
\end{aligned}
$$

When $N$ is even the parity constraint implies $a_{\mathbf{r}} = 0$. When $N$ is odd, the two critical points contribute equally to the asymptotics, leading to

$$a_{\mathbf{r}} = \frac{\left(y_0^{-\hat{r}} z_0^{\hat{r}-1}\right)^N}{\pi\sqrt{\hat{r}}\sqrt{1 - \hat{r}}(1 + 2\hat{r})} N^{-2}.$$

For example, when $\hat{\mathbf{r}} = (1/11, 10/11)$, one obtains $a_{N/11, 10N/11} \approx \frac{2.1792^N}{0.3397\pi} N^{-2}$ for $N$ an odd multiple of 11. When $\hat{r} = 0$, we are counting binary trees. The number of binary trees with $N$ nodes is the $((N - 1)/2)$-th Catalan number. We can see that the exponential growth rate of $a_{\mathbf{r}}$ here agrees with that in Section 5.1 in this case, both yielding $(2 + o(1))^N$.

# References

[BMP19]  Y. Baryshnikov, S. Melczer, and R. Pemantle. Asymptotics of multivariate sequences in the presence of a lacuna. *Preprint, 24 pages*, 2019.

[BP11]    Y. Baryshnikov and R. Pemantle. Asymptotics of multivariate sequences, part III: quadratic points. *Advances in Mathematics*, 228(6):3127–3206, 2011.

[BV13]    M. Bóna and A. Vince. The number of ways to assemble a graph. In *2013 Proceedings of the Tenth Workshop on Analytic Algorithmics and Combinatorics (ANALCO)*, pages 8–17. SIAM, 2013.

[Cha02]    Cyril Chabaud. *Séries génératrices algébriques : asymptotique et applications combinatoires.* PhD thesis, Université Pierre et Marie Curie (UPMC), 2002.

[Eli21]    S. Elizalde. The degree of symmetry of lattice paths. *Annals of Combinatorics*, 25(4):877–911, 2021.

[FO90]    P. Flajolet and A. M. Odlyzko. Singularity analysis of generating functions. *SIAM J. Discrete Math.*, 3(2):216–240, 1990.

[FS09]    P. Flajolet and R. Sedgewick. *Analytic combinatorics.* Cambridge University Press, 2009.

[Fur67]    H. Furstenberg. Algebraic functions over finite fields. *J. Algebra*, 7:271–277, 1967.

[GKZ08]    I. M. Gel'fand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, resultants and multi-dimensional determinants.* Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2008.

[GMRW22]    T. Greenwood, S. Melczer, T. Ruza, and M. Wilson. Asymptotics of coefficients of algebraic series via embedding into rational series. *Preprint*, 2022.

[Gre18]    T. Greenwood. Asymptotics of bivariate analytic functions with algebraic singularities. *J. Comb. Theory A*, 153:1–30, 2018.

[Hat02]    A. Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[HS22]    S. Hossein and S. Starr. Generating function for pinsky's combinatorial second moment formula for the generalized ulam problem. *arXiv*, Math.PR/2301.00125, 2022.

[Mel21]    S. Melczer. *An Invitation to Analytic Combinatorics: From One to Several Variables.* Texts & Monographs in Symbolic Computation. Springer International Publishing, 2021.

[Mik04]    G. Mikhalkin. Amoebas of algebraic varieties and tropical geometry. In *Different faces of geometry*, volume 3 of *Int. Math. Ser. (N. Y.)*, pages 257–300. Kluwer/Plenum, New York, 2004.

[Mis19]    Marni Mishna. *Analytic combinatorics: a multidimensional approach.* CRC Press, Boca Raton, 2019.

[MW19]    S. Melczer and M. Wilson. Higher dimensional lattice walks: connecting combinatorial and analytic behavior. *SIAM J. Disc. Math.*, 33(4):2140–2174, 2019.

[PV19]    F. Pinna and C. Viola. The saddle-point method in $\mathbf{c}^n$ and the generalized Airy functions. *Bull. Math. Soc. France*, 147(2):211–257, 2019.

[PW02]    R. Pemantle and M. C. Wilson. Asymptotics of multivariate sequences. I. Smooth points of the singular variety. *J. Combin. Theory Ser. A*, 97(1):129–161, 2002.

[PW04]   R. Pemantle and M. C. Wilson. Asymptotics of multivariate sequences. II. Multiple points of the singular variety. *Combin. Probab. Comput.*, 13(4-5):735–761, 2004.

[PW08]   R. Pemantle and M. C. Wilson. Twenty combinatorial examples of asymptotics derived from multivariate generating functions. *SIAM Rev.*, 50(2):199–272, 2008.

[PW13]   R. Pemantle and M. Wilson. *Analytic Combinatorics in Several Variables*, volume 340 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, New York, 2013.

[PWM23]  R. Pemantle, M. Wilson, and S. Melczer. *Analytic Combinatorics in Several Variables*, volume 340 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, New York, second edition, 2023.

[RW02]   A. Raichev and M. C. Wilson. A new approach to asymptotics of Maclaurin coefficients of algebraic functions. *ArXiv e-prints*, 2012-02.

[RW07]   A. Raichev and M. C. Wilson. A new method for computing asymptotics of diagonal coefficients of multivariate generating functions. In *2007 Conference on Analysis of Algorithms, AofA 07*, Discrete Math. Theor. Comput. Sci. Proc., AH, pages 439–449. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2007.

[Saf00]  K. V. Safonov. On power series of algebraic and rational functions in $\mathbf{C}^n$. *J. Math. Anal. Appl.*, 243(2):261–277, 2000.

[Sta97]  R. P. Stanley. *Enumerative combinatorics. Vol. 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1997. With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original.

:

# A   Appendix: orientation

{sec:orientati

We recall those hypotheses from Theorem 3.4 that will be needed for sign determination. The first two follow from the standing hypotheses preceding the theorem.

real   The defining polynomial $P = P(\mathbf{z}, f)$ is real.

branches   All roots of the restriction of $P$ to the $F$ axis are simple (i.e. $\operatorname{discr}(\mathbf{0}) \neq 0$), and one of them defines the generating function $F$ we consider.

boundary   The polydisk of convergence of $F$ has radii $(e^{p_1}, \ldots, e^{p_d})$, and the corresponding point $\mathbf{p} = (p_1, \ldots, p_d)$ lies on the boundary of the component of the complement to the amoeba of the discriminant discr (and is inside the same component of the complement to the amoeba of the leading coefficient of $P$, as image of the small torus).

simple   The vertical fiber $\{\mathbf{z} = \mathbf{z}_*\}$ has a simple tangency to the variety $\{P = 0\}$ at the real point $(\mathbf{z}_*, f_*)$; here $\mathbf{z}_* = (e^{p_1}, \ldots, e^{p_d})$ and $f_* = \lim_{\mathbf{z} \to \mathbf{z}_*} f_j(\mathbf{z})$.

convex  The logarithmic Gauss map $\mathbf{br} \to \mathbb{CP}^{d-1}$ is strongly convex, meaning quadratically nondegenerate at $\mathbf{z}_*$ (here $\mathbf{br} = \{\text{discr} = 0\}$).

If $(\mathbf{z}_*, f_*) \in \widetilde{\mathcal{V}}$ is a critical point of the phase $|\mathbf{z}^{\mathbf{r}}|$, it follows that near that critical point, the defining polynomial $P$ can be represented as

$$P = \hat{\mathbf{r}} \cdot (\mathbf{z} - \mathbf{z}_*) + q(\mathbf{z} - \mathbf{z}_*, f - f_*) \tag{A.1}$$ {eq:eq}

where $q = Q + R_3$, where $Q$ is a real quadratic form, $R_k$, here and throughout, denotes a remainder term vanishes to order $k$ in $f - f_*$ and $\mathbf{z} - \mathbf{z}_*$.

{lem:orient ro

**Lemma A.1.** *Assume the hypotheses of the theorem. Let $F$ be algebraic function, and $\hat{\mathbf{r}}$ is the logarithmic gradient of the discriminant at $\mathbf{z}_*$. Then*

1. *The lift of a small d-torus $\mathbf{T}_\epsilon$ around the origin to the branch of $\widetilde{\mathcal{V}}$ defining $f = f_j$ is homologous (can be in fact deformed within the space of embeddings into $\widetilde{\mathcal{V}}$) to a d-torus $\mathbf{T}_* \in \widetilde{\mathcal{V}}$ passing through $\mathbf{z}_*$ so that the restriction of the phase function $|\mathbf{z}^{\hat{\mathbf{r}}}|$ to $\mathbf{T}_*$ attains its global maximum at $(\mathbf{z}_*, f_*)$, and is Morse there.*

2. *Let $\eta$ be a $(d-1)$ holomorphic form defined in a vicinity of $\mathbf{z}_*$, such that $d\phi \wedge \eta = \omega$ , where $\omega := \prod_k \frac{1}{2\pi i} \frac{dz_k}{z_k}$. Then $df \wedge \eta$ defines the orientation on $\mathbf{T}_*$ (near $(\mathbf{z}_*, f_*)$) consistent with the orientation inherited from the the orientation of $\mathbf{T}_\epsilon$ if the real branch of $f$ approaches $f_*$ from below, and with opposite orientation if the real branch of $f$ approaches $f_*$ from above.*

*Remark.* The second claim of the proposition seems to depend unexpectedly on the orientation of the $f$ axis; we remark that the integrated form includes a factor $\partial q / \partial f$, which flips sign together with $df$, leaving the final integral invariant with respect to such flips.

PROOF:   To prove the first claim, we can consider a radii increasing homotopy of the tori in the $\mathbf{z}$-space to the torus whose radii given by $\exp(\mathbf{p})$. During this expansion, up to the last point, all the tori are disjoint from $\mathbf{br}$, hence, over the domain of this deformation, the projection $\widetilde{\mathcal{V}}_* \to \mathbb{C}_*^d$ is a covering, and the homotopy can be lifted to $\widetilde{\mathcal{V}}$, producing at time $t = 1$ the torus $\tilde{T}'$. For $\varepsilon > 0$ sufficiently small, we will need to analyze the time-$(1 - \varepsilon)$ torus, which we denote by $\mathbf{T}_-$.

We switch to local exponential coordinate chart centered at $\mathbf{p}_*$: we chose the real at $\mathbf{p}_*$ branch of the logarithm, and denote all variables in log space by upper case letters, except for $\mathbf{p}$. Thus, $z_k = \exp(p_k + Z_k)$, with $Z_k = X_k + iY_k$. We do the same with the $f$ coordinate: $f = f_* \exp(G + iH)$.

One can always perform a real translation and linear volume preserving transformation on $\mathbf{Z}$ so that in the new coordinates $(W_1, \ldots, W_d)$ we have $W_1 = \hat{\mathbf{r}} \cdot \mathbf{Z}$. Further, we can choose the $W_1$ axis and the deformation of tori so that it ends with a short segment on the negative half-axis $W_1$.

All these transformations together result in the local description of the variety $\mathcal{V}$ in the new coordinates $(F, W_1, \mathbf{W})$ (here $\mathbf{W} = (W_2, \ldots, W_d)$) as

$$\mathcal{V} = \{W_1 = \phi(F, \mathbf{W})\}, \tag{A.2}$$ {eq:local}

with $\phi$ vanishing at the origin to order 2.

The *[simple]* condition implies that we can expand the quadratic part $\phi_2$ of $\phi$ as (the reason for the signs will become clearer later):

$$\phi_2 = -aF^2 + 2F\mathbf{b} \cdot \mathbf{W} - q(\mathbf{W}).$$

Here, $\mathbf{b}$ is a real covector, and $q$ is a real quadratic form.

By the *[branches]* condition, two real branches of $f$ merge at $\mathbf{z}_*$ when $\mathbf{z}$ follows the homotopy from the origin to $e^{\mathbf{p}_*}$, implying that for $\epsilon > 0$ there are real solutions to $-\epsilon = -aF^2 + R_3$, obtained by setting $\mathbf{W} = 0$ in (A.2), so that $a > 0$.

As the logarithmic coordinate change acts in each coordinate independently, the discriminant can be computed in logarithmic coordinates. This can be accomplished by eliminating $F$ from an equation that tracks sufficiently many terms to give us the leading (quadratic) expression for $W_1$ in terms of the other variables:

$$0 = \frac{\partial \phi}{\partial F} = -2aF + 2\mathbf{b} \cdot \mathbf{W} + R_2; W_1 = \phi(F, \mathbf{W}).$$

This results in

$$W_1 = \frac{(\mathbf{b} \cdot \mathbf{W})^2}{a} - q(\mathbf{W}) + R_3$$

as the local equation defining the discriminant variety $\mathbf{br}$.

The real part of $\mathbf{br}$ projects under the log mapping to the contour of the amoeba of $\mathrm{discr}(P)$. The *[convex]* condition implies that the contour near the origin is smooth, coincides with the boundary of the amoeba, and is *quadratically convex*, hence the quadratic form $\dfrac{(\mathbf{b} \cdot \mathbf{W})^2}{a} - q(\mathbf{W})$ is negative definite.

After these preliminaries, we can look at the lift of the torus $\mathbf{T}_-$, i.e., the intersection of the preimage of the torus in $\mathbb{C}^d$ under the projection along $F$ with the variety $\widetilde{\mathcal{V}}$. In our local log coordinates, where we denote the real and imaginary parts as $W_k = U_k + iV_k, k = 2, \ldots, d$, this preimage corresponds to setting $W_1 = -\epsilon + iV_1, W_k = iV_k, k \geq 2$, and $F = G + iH$.

Expanding the terms above, we arrive at

$$-\epsilon + iV_1 = -a(G^2 - H^2) - 2aiGH + 2G\mathbf{b} \cdot i\mathbf{V} + 2iH\mathbf{b} \cdot i\mathbf{V} + q(\mathbf{V}) + R_3$$

where the indices of $\mathbf{V}$ run from 2 to $d$, that is, $\mathbf{V} = (V_2, \ldots, V_d)$. The real part of this equation is

$$aG^2 = \epsilon + aH^2 - 2H\mathbf{b} \cdot \mathbf{V} + q(\mathbf{V}) = \epsilon + a(H - \mathbf{b} \cdot \mathbf{V}/a)^2 - (\mathbf{b} \cdot \mathbf{V})^2/a + q(\mathbf{V}) + R_3.$$

The quadratic form on right hand side above is positive definite because $(\mathbf{b} \cdot \mathbf{V})^2/a - q(\mathbf{V})$ is negative definite. Hence, for small $\varepsilon > 0$, the equation above defines a hypersurface diffeomorphic to a two-sheeted hyperboloid in the $(G, H, \mathbf{V})$ space. The sign of $G$ on the branch below the merge point, is negative; above, positive. Projection of this surface to $H, \mathbf{V}$ gives coordinates on the sheet, so that $G$ becomes a function of $H, \mathbf{V}$.

Now, one can also express

$$V_1 = -2aGH + 2G\mathbf{b} \cdot \mathbf{V} + R_3.$$

This shows that locally, the chain $\mathbf{T}_-$ can be locally coordinatized by $H, \mathbf{V}$.

In our new coordinates the real part of the phase is $U_1$; the torus $\mathbf{T}_-$ is situated at the level set of the phase. Outside a vicinity of $(\mathbf{z}_*, f_*)$ the gradient of the phase $\phi$ is non-vanishing, and, by compactness, one can

29

deform the chain there to the zero sublevel set of $\Re\{\phi\}$. *Within* the vicinity of the critical point, one can use the coordinatization by $H, \mathbf{V}$ to deform the chain to the $d$-space spanned by $H, \mathbf{V}$: along that subspace,

$$U_1 = Re(\phi(F, \mathbf{W})) = aH^2 + q(\mathbf{V}) + R_3$$

showing $U_1$ to be Morse and positive definite.

To compute the orientation, we represent $\omega = (2\pi i)^{-d} dW_1 \wedge \cdots \wedge dW_d$ in local logarithmic coordinates. On the variety $\widetilde{\mathcal{V}}$ one has, using (A.2)

$$\omega = \frac{1}{(2\pi i)^d} \frac{\partial \phi}{\partial F} dF \wedge d\mathbf{W}$$

where $d\mathbf{W} := dW_2 \wedge \cdots \wedge dW_d$.

Coordinatizing the variety $\widetilde{\mathcal{V}}$ locally by $F, \mathbf{W}$, we obtain

$$\omega = \frac{-2}{(2\pi i)^d} (aF - \mathbf{b} \cdot \mathbf{W}) dF \wedge d\mathbf{W}\,.$$

At the point where $W_1 = -\epsilon$ and $\mathbf{W} = \mathbf{0}$, this reduces to

$$\omega = \frac{-2}{(2\pi)^d} (aG)\, dH \wedge d\mathbf{V}\,.$$

Thus $\omega$ is a *positive* multiple of $dF \wedge d\mathbf{W}$ on $\mathbf{T}_-$ on the branch where $G < 0$, i.e. $f < f_*$, and *negative* where $f > f_*$. Equivalently, the orientation is given by $dH \wedge dY_2 \wedge \ldots \wedge dY_d$ on the lower branch of $f$, and is opposite that on the upper branch. $\qquad\square$