Introduction to Robotics
Lecture 14: Lagrangian dynamics
Dynamics of open chains

It is often implicitly assumed that the robots' links have negligible mass, at least compared to the actuation power of their actuators. In this case, the kinematic approach to motion describes the actual physical system relatively well. We look in this lecture into the effects of non-negligible masses, and thus inertia, on the dynamics of robots.

**Inverse dynamics:** determine the torques yielding a given trajectory \((\theta, \dot{\theta}, \ddot{\theta})\):

\[
\tau = M(\theta)\ddot{\theta} + h(\theta, \dot{\theta})
\]

**Forward dynamics:** determines the trajectory \(\theta(t)\) given \(\tau\):

\[
\ddot{\theta} = M^{-1}(\theta)(\tau - h(\theta, \dot{\theta})).
\]

The trajectory is a solution of the above ODE with appropriate initial state.

\(M(\theta)\) is called the **mass matrix**.

Inverse: Given torques \(\tau\), find \(\theta\) (It is most important for mobile robots).
We first review the Lagrangian approach to determine the dynamics of a rigid body.

Denote by \( \mathbf{q} \in \mathbb{R}^n \) the so-called generalized coordinates of the system. These are a set of coordinates describing its state.

From the generalized coordinates, we define the generalized forces \( \mathbf{f} \in \mathbb{R}^n \). These are the forces on the system that "act" on the generalized coordinates.

The pair needs to be consistently chosen so that the power dissipated by the system is \( \mathbf{f}^\top \dot{\mathbf{q}} \).

\[ \begin{align*}
\theta &\rightarrow \mathbf{q} \quad \text{(traditional, \( (q, \dot{q}) \)-coordinates)} \\
\mathbf{f} &\leftarrow \text{dual, appears as } \mathbf{f}^\top \dot{\mathbf{q}}
\end{align*} \]
Lagrangian dynamics: point mass

Mathematically, generalized forces are the dual to the (derivatives) of the generalized coordinates. We do not go into these details in this course.

Denote by $K(q, \dot{q})$ the kinetic energy of the system, and by $P(q)$ its potential energy. (Recall that potential energy does not depend on $\dot{q}$). The Lagrangian of the system is defined as

$$L(q, \dot{q}) = K(q, \dot{q}) - P(q)$$

From the Lagrangian of the system, we obtain the equations of motion through the principle of least action, which yields the Euler-Lagrange equations:

$$f = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}.$$

Note:
Lagrangian dynamics: point mass

Consider a particle of mass $m$ constrained to move on the vertical line.

A generalized coordinate is its height $x \in \mathbb{R}$.

Suppose that an external force $f$ is applied on it, and force due to gravity is given by $mg$. The Lagrangian is then

$$L(x, \dot{x}) = K(x, \dot{x}) - P(x) = \frac{1}{2} m \dot{x}^2 - mgx.$$

The equations of motion are given by

$$f = \frac{d}{dt}(m\ddot{x}) - (-mg) = m\ddot{x} + mg.$$

We obtain the same equations of motion using Newton's $f = ma$ law.

What is remarkable, is invariance with coordinate change $x = s^2$.

$\frac{1}{2} m \cdot 4s^2 \dot{s}^2 - wgs^2 = 2ws^2 \dot{s}^2 - wgs^2$

$\frac{4w^2s^2\dot{s}^2}{4} - 4wss^2 + 2wgs = 0$

$2s^2 + \dot{s}^2 = -g$
Lagrangian dynamics: 2R open chain

Consider a 2R open chain, with links of masses \( m_1, m_2 \) respectively. To simplify things, we assume that the masses are concentrated at the ends of links.

We take the joint positions \((\theta_1, \theta_2)\) for generalized coordinates, and \((\tau_1, \tau_2)\), the torques applied at the joints, as generalized forces. Note that \(\tau^\top \dot{\theta}\) is the power dissipated by the torques.

We now need to derive the Lagrangian. To this end, we need the position and velocity of the masses.

Rather than express \(L\) in terms of \(x, y, \dot{x}, \dot{y}, \ldots\), work directly in \(\theta_1, \theta_2\).
Lagrangian dynamics: 2R open chain

\[
\sum x_1 y_1 = \sum L_1 \cos \theta_1 \cos \theta_1 \sin \theta_1 \prod \text{ and } \\
\sum \dot{x}_1 \dot{y}_1 = \sum \dot{L}_1 \sin \theta_1 \cos \theta_1 \sin \theta_1 \prod \\
\sum x_2 y_2 = \sum L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) \prod \text{ and } \\
\sum \dot{x}_2 \dot{y}_2 = \sum -L_1 \sin \theta_1 - L_2 \sin(\theta_1 + \theta_2) \prod \text{ and } \\
\sum \dot{x}_2 \dot{y}_2 = \sum L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) \prod \\
\sum \dot{x}_1 \dot{y}_1 = \sum L_1 \sin \theta_1 \cos \theta_1 \sin \theta_1 \prod \\
\sum \dot{x}_1 \dot{y}_1 = \sum -L_1 \sin \theta_1 - L_2 \sin(\theta_1 + \theta_2) \prod \\
\sum \dot{x}_2 \dot{y}_2 = \sum L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) \prod \\
\sum \dot{x}_2 \dot{y}_2 = \sum L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) \prod .
\]

For link 2:
Using the relations of the previous slide, we obtain the kinetic energy of the links:

\[
K_1 = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) = \frac{1}{2} m_1 L_1^2 \dot{\theta}_1^2
\]

\[
K_2 = \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)
\]

\[
= \frac{1}{2} m_2 [(L_2^2 + 2L_1 L_2 \cos \theta_2 + L_2^2) \dot{\theta}_1^2 +
+ 2(L_1^2 + L_1 L_2 \cos \theta_2) \dot{\theta}_1 \dot{\theta}_2 + L_2^2 \dot{\theta}_2^2]
\]

The potential energies of the links are

\[
P_1 = m_1 g y_1 = m g L_1 \sin \theta_1
\]

\[
P_2 = m_2 g y_2 = m_2 g (L_1 \sin \theta_1 + L_2 \sin(\theta_1 + \theta_2))
\]
Lagrangian dynamics: 2R open chain

The equations of motion are \( \tau_i = \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i}, \ i = 1, 2. \)

This yields here

\[
\begin{align*}
\tau_1 &= \left( m_1 L_1^2 + m_2 (L_1^2 + 2L_1 L_2 \cos \theta_2 + L_2^2) \right) \ddot{\theta}_1 \\
&\quad + m_2 (L_1 L_2 \cos \theta_2 + L_2^2) \ddot{\theta}_2 - m_2 L_1 L_2 \sin \theta_2 (2 \dot{\theta}_1 \dot{\theta}_2 + \dot{\theta}_2^2) \\
&\quad + (m_1 + m_2) L_1 g \cos \theta_1 + m_2 g L_2 \cos (\theta_1 + \theta_2), \\
\tau_2 &= m_2 (L_1 L_2 \cos \theta_2 + L_2^2) \ddot{\theta}_1 + m_2 L_2^2 \ddot{\theta}_2 + m_2 L_1 L_2 \theta_1^2 \sin \theta_2 \\
&\quad + m_2 g L_2 \cos (\theta_1 + \theta_2).
\end{align*}
\]
Lagrangian dynamics: 2R open chain

We can write the above equations as

$$\tau = M(\theta)\ddot{\theta} + c(\theta, \dot{\theta}) + g(\theta)$$

with the definitions

$$M(\theta) = \begin{bmatrix} m_1 L_1^2 + m_2(L_1^2 + 2L_1 L_2 \cos \theta_2 + L_2^2) & m_2(L_1 L_2 \cos \theta_2 + L_2^2) \\ m_2(L_1 L_2 \cos \theta_2 + L_2^2) & m_2 L_2^2 \end{bmatrix}$$

$$c(\theta, \dot{\theta}) = \begin{bmatrix} -m_2 L_1 L_2 \sin \theta_2 (2\dot{\theta}_1 \dot{\theta}_2 + \dot{\theta}_2^2) \\ m_2 L_1 L_2 \dot{\theta}_1^2 \sin \theta_2 \end{bmatrix},$$

$$g(\theta) = \begin{bmatrix} (m_1 + m_2)L_1 g \cos \theta_1 + m_2 g L_2 \cos(\theta_1 + \theta_2) \\ m_2 g L_2 \cos(\theta_1 + \theta_2) \end{bmatrix},$$

let $M = m_1 L_1^2 + m_2 L_1^2 \sin^2 \theta_2 > 0$ if $L_1, m_i > 0$. 
The matrix $M(\theta)$ is symmetric and positive definite. It is called the mass matrix.

The vector $c(\theta, \dot{\theta})$ contains the centripetal and Coriolis forces/torques, and $g(\theta)$ contains the gravitational forces/torques.

We could have obtained the same equations again from $f = ma$

\[
f_1 = \begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{z1} \end{bmatrix} = m_1 \begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{z}_1 \end{bmatrix} = m_1 \begin{bmatrix} -L_1 \dot{\theta}_1^2 c_1 - L_1 \ddot{\theta}_1 s_1 \\ -L_1 \dot{\theta}_1^2 s_1 + L_1 \ddot{\theta}_1 c_1 \\ 0 \end{bmatrix},
\]

\[
f_2 = m_2 \begin{bmatrix} -L_1 \dot{\theta}_1^2 c_1 - L_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 c_{12} - L_1 \ddot{\theta}_1 s_1 - L_2 (\ddot{\theta}_1 + \ddot{\theta}_2) s_{12} \\ -L_1 \dot{\theta}_1^2 s_1 - L_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 s_{12} + L_1 \ddot{\theta}_1 c_1 + L_2 (\ddot{\theta}_1 + \ddot{\theta}_2) c_{12} \\ 0 \end{bmatrix}
\]

Note that since $(\hat{x}, \hat{y})$ is an inertial frame, we have equations $\ddot{x} = \cdots, \ddot{y} = \cdots$.

The frame $(\hat{\theta}_1, \hat{\theta}_2)$ is not inertial, hence $\ddot{\theta} = 0$ does not mean there is zero acceleration (i.e., $\ddot{x}, \ddot{y}$ are not necessarily zero when $\ddot{\theta}$ is.).
Lagrangian dynamics: 2R open chain

A zero acceleration in a non-inertial frame does not imply a zero acceleration in an inertial frame.

Consider the arm in position \((\theta_1, \theta_2) = (0, \pi/2)\). Assuming \(\ddot{\theta} = 0\), we have

\[
\begin{bmatrix}
\dot{x}_2 \\
\dot{y}_2
\end{bmatrix}
= \begin{bmatrix}
-L_1 \dot{\theta}_1^2 \\
-L_2 \dot{\theta}_1^2 - L_2 \dot{\theta}_2^2
\end{bmatrix} + \begin{bmatrix}
0 \\
-2L_2 \dot{\theta}_1 \dot{\theta}_2
\end{bmatrix}
\]

\[
\text{centripetal} \quad \text{Coriolis}
\]

Quadratic terms \(\dot{\theta}_i^2\) are called centripetal terms, the mixed quadratic terms \(\dot{\theta}_1 \dot{\theta}_2\) the Coriolis terms.

\[
\varphi = \arctan \frac{t}{r} - \varphi
\]

\[
r = \sqrt{1 + t^2}
\]
Lagrangian dynamics: 2R open chain

If $\dot{\theta}_2 = 0$, no Coriolis and centrifugal accel. is 
$(-L_1 \dot{\theta}_1^2, -L_2 \dot{\theta}_1^2)$. Similarly, $\dot{\theta}_1 = 0$, no Coriolis and 
centrifugal accel. is $(0, -L_2 \dot{\theta}_2^2)$.

These accelerations keep the mass rotating around 
the joints 1 and 2 respectively.

The Coriolis force appears if both $\dot{\theta}_i$ are non-zero. 
Note that its sign depends on the signs of the $\theta_i$’s.

('[frame at joints]').
For a general open chain with \( n \) links, we take the link angles \( \theta_i \) as generalized coordinates and the corresponding torques \( \tau_i \) as generalized forces. The kinetic energy can always be written as

\[
K(\theta, \dot{\theta}) = \dot{\theta}^T M(\theta) \dot{\theta}
\]

for an appropriately defined mass matrix \( M(\theta) \).

The dynamics equation are then

\[
\tau_i = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_i} \right) - \frac{\partial L}{\partial \theta_i}
\]

for \( L = K - P \) with \( P(\theta) \) the potential energy of the system.

Usually, \( K \) is a quadratic function of \( \dot{\theta}_i \), but with coefficients depending on \( \theta_i \).
Lagrangian dynamics for general chains

Written explicitly, we have

$$\tau_i = \sum_{j=1}^{n} m_{ij}(\theta)\dot{\theta}_j + \sum_{j,k=1}^{n} \Gamma_{ijk}(\theta)\dot{\theta}_j \dot{\theta}_k + \frac{\partial P}{\partial \theta_i}$$

where

$$\Gamma_{ijk}(\theta) = \frac{1}{2} \left( \frac{\partial m_{ij}}{\partial \theta_k} + \frac{\partial m_{ik}}{\partial \theta_j} - \frac{\partial m_{jk}}{\partial \theta_i} \right)$$

are the Christoffel’s symbols of the first kind.

This dynamics is also written as

$$\tau = M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + g(\theta),$$

where $M$ is the mass matrix, and $C(\theta, \dot{\theta}) \in \mathbb{R}^{n \times n}$ is the matrix with entries

$$c_{ij} = \sum_{k=1}^{n} \Gamma_{ijk}(\theta)\dot{\theta}_k.$$ 

It is called the Coriolis matrix.