Introduction to Robotics
Lecture 14: Lagrangian dynamics
Dynamics of open chains

It is often implicitly assumed that the robots' links have negligible mass, at least compared to the actuation power of their actuators. In this case, the kinematic approach to motion describes the actual physical system relatively well. We look in this lecture into the effects of non-negligible masses, and thus inertia, on the dynamics of robots.

**Inverse dynamics**: determine the torques yielding a given trajectory \((\theta, \dot{\theta}, \ddot{\theta})\):

\[
\tau = M(\theta)\ddot{\theta} + h(\theta, \dot{\theta})
\]

**Forward dynamics**: determines the trajectory \(\theta(t)\) given \(\tau\):

\[
\dot{\theta} = M^{-1}(\theta)(\tau - h(\theta, \dot{\theta})).
\]

The trajectory is a solution of the above ODE with appropriate initial state.

\(M(\theta)\) is called the *mass matrix*.
Lagrangian dynamics

We first review the Lagrangian approach to determine the dynamics of a rigid body.

Denote by $q \in \mathbb{R}^n$ the so-called generalized coordinates of the system. These are a set of coordinates describing its state.

From the generalized coordinates, we define the generalized forces $f \in \mathbb{R}^n$. These are the forces on the system that “act” on the generalized coordinates.

The pair needs to be consistently chosen so that the power dissipated by the system is $f^\top \dot{q}$. 

Lagrangian dynamics: point mass

Mathematically, generalized forces are the dual to the (derivatives) of the generalized coordinates. We do not go into these details in this course.

Denote by \( K(q, \dot{q}) \) the kinetic energy of the system, and by \( P(q) \) its potential energy. (Recall that potential energy does not depend on \( \dot{q} \)). The Lagrangian of the system is defined as

\[
L(q, \dot{q}) = K(q, \dot{q}) - P(q)
\]

From the Lagrangian of the system, we obtain the equations of motion through the principle of least action, which yields the Euler-Lagrange equations:

\[
f = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}.
\]
Lagrangian dynamics: point mass

Consider a particle of mass $m$ constrained to move on the vertical line.

A generalized coordinate is its height $x \in \mathbb{R}$.

Suppose that an external force $f$ is applied on it, and force due to gravity is given by $mg$. The Lagrangian is then

$$L(x, \dot{x}) = K(x, \dot{x}) - P(x) = \frac{1}{2} mx^2 - mgx.$$

The equations of motion are given by

$$f = \frac{d}{dt}(m\dot{x}) - (-mg) = m\ddot{x} + mg.$$

We obtain the same equations of motion using Newton's $f = ma$ law.

$$\frac{d}{dt}(4ms^2\dot{s}) = 4m (s\ddot{s}) = 4m (2s\dot{s}^2 + s^2 \ddot{s}) = 4ms^2 - 2mgs$$
Lagrangian dynamics: point mass

Consider a particle of mass $m$ constrained to move on the vertical line.

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We obtain the same equations of motion using Newton's $f = ma$ law.
Consider a 2R open chain, with links of masses $m_1, m_2$ respectively. To simplify things, we assume that the masses are concentrated at the ends of links.

We take the joint positions $(\theta_1, \theta_2)$ for generalized coordinates, and $(\tau_1, \tau_2)$, the torques applied at the joints, as generalized forces. Note that $\tau^T \dot{\theta}$ is the power dissipated by the torques.

We now need to derive the Lagrangian. To this end, we need the position and velocity of the masses.

\[ K = \frac{m_1}{2} (\dot{x}_1^2 + \dot{y}_1^2) + \frac{m_2}{2} (\dot{x}_2^2 + \dot{y}_2^2) \]

\[ P = m_1 L_1 \sin \theta \, \dot{\theta}, g + m_2 (L_1 \dot{x}_1 + L_2 \dot{x}_2) \]
Lagrangian dynamics: 2R open chain

\[
\begin{align*}
\sum x_1 y_1 &= \sum L_1 \cos \theta_1 L_1 \sin \theta_1 \\
\prod &= \prod \\
\end{align*}
\]

and

\[
\begin{align*}
\sum \dot{x}_1 \dot{y}_1 &= \sum \dot{L}_1 \sin \theta_1 \dot{L}_1 \cos \theta_1 \\
\prod &= \prod \\
\end{align*}
\]

For link 2:

\[
\begin{align*}
x_2 &= \begin{bmatrix} L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) \\ L_1 \sin \theta_1 + L_2 \sin(\theta_1 + \theta_2) \end{bmatrix} \\
y_2 &= \begin{bmatrix} L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) \\ L_1 \sin \theta_1 + L_2 \sin(\theta_1 + \theta_2) \end{bmatrix}
\end{align*}
\]

and

\[
\begin{align*}
x_2 &= \begin{bmatrix} -L_1 \sin \theta_1 - L_2 \sin(\theta_1 + \theta_2) \\ L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \\
y_2 &= \begin{bmatrix} -L_1 \sin \theta_1 - L_2 \sin(\theta_1 + \theta_2) \\ L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\prod &= \prod \\
\end{align*}
\]
Using the relations of the previous slide, we obtain the kinetic energy of the links:

\[
K_1 = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) = \frac{1}{2} m_1 L_1^2 \dot{\theta}_1^2 \\
K_2 = \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) \\
= \frac{1}{2} m_2 [(L_2^2 + 2L_1L_2 \cos \theta_2 + L_2^2) \dot{\theta}_1^2 + \\
+ 2(L_2^2 + L_1L_2 \cos \theta_2) \dot{\theta}_1 \dot{\theta}_2 + L_2^2 \dot{\theta}_2^2]
\]

The potential energies of the links are

\[
P_1 = m_1 g y_1 = mgL_1 \sin \theta_1 \\
P_2 = m_2 g y_2 = m_2 g (L_1 \sin \theta_1 + L_2 \sin(\theta_1 + \theta_2))
\]
The equations of motion are \( \tau_i = \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i}, \ i = 1, 2. \)

This yields here

\[
\tau_1 = (m_1 L_1^2 + m_2 (L_1^2 + 2L_1L_2 \cos \theta_2 + L_2^2)) \ddot{\theta}_1 \\
+ m_2 (L_1 L_2 \cos \theta_2 + L_2^2) \ddot{\theta}_2 - m_2 L_1 L_2 \sin \theta_2 (2\dot{\theta}_1 \dot{\theta}_2 + \dot{\theta}_1^2) \\
+ (m_1 + m_2) L_1 g \cos \theta_1 + m_2 g L_2 \cos(\theta_1 + \theta_2),
\]

\[
\tau_2 = m_2 (L_1 L_2 \cos \theta_2 + L_2^2) \ddot{\theta}_1 + m_2 L_2^2 \ddot{\theta}_2 + m_2 L_1 L_2 \dot{\theta}_1^2 \sin \theta_2 \\
+ m_2 g L_2 \cos(\theta_1 + \theta_2).
\]
Lagrangian dynamics: 2R open chain

We can write the above equations as

\[ \tau = M(\theta)\ddot{\theta} + c(\theta, \dot{\theta}) + g(\theta) \]

with the definitions

\[ M(\theta) = \begin{bmatrix} m_1L_1^2 + m_2(L_1^2 + 2L_1L_2 \cos \theta_2 + L_2^2) & m_2(L_1L_2 \cos \theta_2 + L_2^3) \\ m_2(L_1L_2 \cos \theta_2 + L_2^3) & m_2L_2^2 \end{bmatrix} \]

\[ c(\theta, \dot{\theta}) = \begin{bmatrix} -m_2L_1L_2 \sin \theta_2(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \\ m_2L_1L_2 \dot{\theta}_1^2 \sin \theta_2 \end{bmatrix} \]

\[ g(\theta) = \begin{bmatrix} (m_1 + m_2)L_1g \cos \theta_1 + m_2gL_2 \cos(\theta_1 + \theta_2) \\ m_2gL_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \]

\[ \det M = m_1m_2L_1L_2^2 + m_2^2L_1^2L_2^2 \sin^2 \theta_2 > 0 \]
Lagrangian dynamics: 2R open chain

The matrix $M(\theta)$ is symmetric and positive definite. It is called the mass matrix.

The vector $c(\theta, \dot{\theta})$ contains the centripetal and Coriolis forces/torques, and $g(\theta)$ contains the gravitational forces/torques.

We could have obtained the same equations again from $f = ma$

Note that since $(\hat{x}, \hat{y})$ is an inertial frame, we have equations $\ddot{x} = \cdots, \ddot{y} = \cdots$.

The frame $(\hat{\theta}_1, \hat{\theta}_2)$ is not inertial, hence $\ddot{\theta} = 0$ does not mean there is zero acceleration (i.e., $\ddot{x}, \ddot{y}$ are not necessarily zero when $\ddot{\theta}$ is.).
Lagrangian dynamics: 2R open chain

A zero acceleration in a non-inertial frame does not imply a zero acceleration in an inertial frame.

Consider the arm in position $(\theta_1, \theta_2) = (0, \pi/2)$. Assuming $\dot{\theta} = 0$, we have

$$\begin{bmatrix} \ddot{x}_2 \\ \ddot{y}_2 \end{bmatrix} = \begin{bmatrix} -L_1 \dot{\theta}_1^2 \\ -L_2 \dot{\theta}_1^2 - L_2 \dot{\theta}_2^2 \end{bmatrix} + \begin{bmatrix} 0 \\ -2L_2 \dot{\theta}_1 \dot{\theta}_2 \end{bmatrix}$$

Quadratic terms $\dot{\theta}_i^2$ are called centripetal terms, the mixed quadratic terms $\dot{\theta}_1 \dot{\theta}_2$ the Coriolis terms.
Lagrangian dynamics: 2R open chain

If $\dot{\theta}_2 = 0$, no Coriolis and centrifugal accel. is $(-L_1 \dot{\theta}_1^2, -L_2 \dot{\theta}_1^2)$. Similarly, $\dot{\theta}_1 = 0$, no Coriolis and centrifugal accel. is $(0, -L_2 \dot{\theta}_2^2)$.

These accelerations keep the mass rotating around the joints 1 and 2 respectively.

The Coriolis force appears if both $\dot{\theta}_i$ are non-zero. Note that its sign depends on the signs of the $\theta_i$'s.
Lagrangian dynamics for general chains

For a general open chain with \( n \) links, we take the link angles \( \theta_i \) as generalized coordinates and the corresponding torques \( \tau_i \) as generalized forces. The kinetic energy can always be written as

\[
K(\theta, \dot{\theta}) = \dot{\theta}^T M(\theta) \dot{\theta}
\]

for an appropriately defined mass matrix \( M(\theta) \).

The dynamics equation are then

\[
\tau_i = \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i}
\]

for \( L = K - P \) with \( P(\theta) \) the potential energy of the system.
Written explicitly, we have

\[ \tau_i = \sum_{j=1}^{n} m_{ij}(\theta) \dot{\theta}_j + \sum_{j,k=1}^{n} \Gamma_{ijk}(\theta) \ddot{\theta}_j \dot{\theta}_k + \frac{\partial P}{\partial \theta_i} \]

where

\[ \Gamma_{ijk}(\theta) = \frac{1}{2} \left( \frac{\partial m_{ij}}{\partial \theta_k} + \frac{\partial m_{ik}}{\partial \theta_j} - \frac{\partial m_{jk}}{\partial \theta_i} \right) \]

are the Christoffel’s symbols of the first kind.

This dynamics is also written as

\[ \tau = M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + g(\theta), \]

where \( M \) is the mass matrix, and \( C(\theta, \dot{\theta}) \in \mathbb{R}^{n \times n} \) is the matrix with entries

\[ c_{ij} = \sum_{k=1}^{n} \Gamma_{ijk}(\theta) \dot{\theta}_k. \]

It is called the Coriolis matrix.
Robot Control
Given sensor readings, how to design actuator torques?
• Denote a desired sequence of joint values by \( \theta_d(t) \), and the actual joint values by \( \theta(t) \). We define the joint error as
\[
\theta_e(t) = \theta_d(t) - \theta(t).
\]

• The differential equation governing the dynamics of \( \theta_e(t) \) is called the error dynamics.

• The commonly used/standard situation to design/analyze a controller performance is the following: we assume that at \( t = 0 \), we have
\[
\theta_e(0) = 1 \text{ and } \dot{\theta}_e(0) = \ddot{\theta}_e(0) = \cdots = 0.
\]

We benchmark controllers’ performances from that initial state.

• We assume here for now that the error dynamics is linear. Robots nonlinear dynamics by far and large, but if the error is small, the linear approximation yields good results.
Error dynamics

- We plot a typical linear error response above.
- We can split the response into a transient response (time during which the dynamics is not negligible) and a steady-state response (when we are almost at equilibrium).
- The steady-state response is characterized by the steady-state error

\[ \theta_{e,ss} := \lim_{t \to \infty} \theta_e(t). \]
The transient response features:

- **overshoot** (how far the error goes past its steady-state):
  \[
  \text{overshoot} = \frac{\theta_{e,\text{min}} - \theta_{e,ss}}{\theta_e(0) - \theta_{e,ss}} \times 100\%
  \]

- **2% settling-time**: the time needed so that
  \[
  \|\theta_e(t) - \theta_{e,ss}\| \leq 0.02(\theta_e(0) - \theta_{e,ss})
  \]
A good error response is characterized by

- small steady-state error
- small overshoot
- short settling time
Linear Error dynamics

- A general linear error dynamics is given by

\[ a_p \theta_e^{(p)} + a_{p-1} \theta_e^{(p-1)} + \cdots + a_2 \dot{\theta}_e + a_1 \dot{\theta}_e + a_0 \theta_e = c(t) \]

- The above equation is called homogeneous if \( c = 0 \) and non-homogeneous if \( c \neq 0 \).

- Introducing new variables, we can write the equation as a first order system of equations:

\[
\begin{align*}
    x_1 & := \theta_e \\
    x_2 & := \dot{x}_1 = \dot{\theta}_e \\
    & \quad \cdots \\
    x_p & := \dot{x}_{p-1} = \theta_e^{(p-1)} \\
    \dot{x}_p & = -a_0/a_p x_1 - a_1/a_p x_2 - \cdots - a_{p-1}/a_p x_p
\end{align*}
\]

LDG of order \( p \)

Readers LDG of order \( p \) as \( p \) LDG's of order 1
Linear Error dynamics

- We can write the previous equation in matrix/vector form: $\dot{x} = Ax$ where $x = (x_1, \ldots, x_p)$ and

$$A = \begin{bmatrix}
    0 & 1 & 0 & \cdots & 0 & 0 \\
    0 & 0 & 1 & \cdots & 0 & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & 0 & \cdots & 1 & 0 \\
    0 & 0 & 0 & \cdots & 0 & 1 \\
    -d_0/a_p & -d_1/a_p & -d_2/a_p & \cdots & -d_{p-1}/a_p & -d_{p-1}/a_p
\end{bmatrix}$$

with $d_i := a_i/a_p$. This matrix is called a companion matrix.

- Recall that the solution of $\dot{x} = Ax$ is $x(t) = e^{At}x(0)$, as we saw earlier in the course. This can be verified by plugging the definition of $e^{At}$ into the differential equation.
Stability of linear systems

- The linear autonomous system \( \dot{x} = Ax \) is said to be \textit{stable} if the real parts of the eigenvalues of \( A \) are \textit{negative}. It is unstable otherwise.
- The state of an unstable system is asymptotically infinite for some initial conditions: there exists \( x_0 \) so that
  \[
  \lim_{t \to \infty} \| x(t) \| = \lim_{t \to \infty} \| e^{At} x_0 \| = \infty
  \]
- It is to verify the stability criterion in case \( A \) is \textit{diagonalizable}, that is \( A \) can be written as \( A = PDP^{-1} \) for some diagonal matrix \( D \) and invertible matrix \( P \). In this case, \( D \) has the eigenvalues of \( A \) in its diagonal and
  \[
  e^{At} = Pe^{Dt} P^{-1}.
  \]

Now recall that \( e^{dt} \), for \( d = a + bi \in \mathbb{C} \) is \( e^{dt} = e^{at} (\cos bt + i \sin bt) \). If \( a > 0 \), \( e^{at} \) grows large. If \( a < 0 \), \( e^{at} \) goes to zero.

\textit{A requirement of any controller: the error dynamics is stable}\n
We now consider the angle $\theta(t)$ at a joint, omitting the index $i$ for now.

If the error dynamics is of **first order**, then it can be written as

$$\dot{\theta}_e(t) + \frac{k}{b} \theta_e(t) = 0$$

You can think of this as a P controller (P is for **proportional**), with parameter $k$: if $\theta_e > 0$, then change $\theta_e$ so that it decreases, hence choose $k < 0$. We come back to this later.

The solution of this equation is

$$\theta_e(t) = e^{-k/b} \theta_e(0).$$

Set $\tau = b/k$. There is no overshoot, and

$$\frac{\theta_e(t)}{\theta_0} = 0.02 = e^{-t/\tau} \Rightarrow t = 3.91 \tau$$