Introduction to Robotics
Week 5: Forward Kinematics
The forward kinematics of a robot refers to the calculation of the position and orientation of its effector frame from its joint coordinates.

Here, simple trigonometry yields

\[
\begin{align*}
x &= L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3) \\
y &= L_1 \sin \theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3) \\
\phi &= \theta_1 + \theta_2 + \theta_3
\end{align*}
\]
Forward kinematics

For more complex 3D mechanisms, a direct analysis as done in previous slide is too onerous.

We describe two systematic, principle ways to perform forward kinematics: **Product of Exponentials (PoE)** and **Denavit-Hartenberg**.

What we can do using previous lectures: attaching frames 1,2,3,4 to the three links and end-effector respectively, and denoting by 0 the reference frame, we need $T_{04}$, which we can obtain as

$$T_{04} = T_{01} T_{12} T_{23} T_{34},$$

and $T_{i(i+1)}$ are easy to derive.
Forward kinematics: using homogeneous transformations

$T_{01} = \begin{bmatrix}
\cos \theta_1 & -\sin \theta_1 & 0 & 0 \\
\sin \theta_1 & \cos \theta_1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$,

$T_{12} = \begin{bmatrix}
\cos \theta_2 & -\sin \theta_2 & 0 & L_1 \\
\sin \theta_2 & \cos \theta_2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$,

$T_{23} = \begin{bmatrix}
\cos \theta_3 & -\sin \theta_3 & 0 & L_2 \\
\sin \theta_3 & \cos \theta_3 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$,

$T_{34} = \begin{bmatrix}
1 & 0 & 0 & L_3 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$.
Forward kinematics: using screw motions

Assume the arm is in “resting position”:
\[ M := T(I, (L_1 + L_2 + L_3, 0, 0)\) \]

Any \( T \) is results of path from some base point.

Revolute joints allow screw motions with zero pitch.
Assume \( \theta_1 = \theta_2 = 0 \). What is the screw motion of joint 3 by its twist in frame 0: \( S_3 = [\omega_3, v_3] \)?

\[ \Theta = \{ [\theta_1, ..., \theta_n] \} \]

Key to the compositionality by applying each next transform at joint that was not affected by previous ones.

Rotation by \( \theta_3 \) around \( \mathbf{e}_3 \).

\[ \begin{bmatrix} \theta_3 \end{bmatrix} \begin{bmatrix} \mathbf{S} \end{bmatrix} = \begin{bmatrix} 0 & -10 & 0 \\ 10 & 0 & -L_3 \\ 0 & 0 & 0 \end{bmatrix} \]
Forward kinematics: using screw motions

How to obtain $\omega_3, v_3$?

Revolute joint: rotation around axis perpendicular to motion and $\omega$ is normalized vector in this direction $\longrightarrow \omega_3 = (0,0,1)^T$.

To find $v_3$, consider a rigid body attached to the joint and following the joint's motion: $v_3$ is the velocity of the point at the origin of frame 0. This speed is $\omega \times (-L_1 - L_2, 0, 0)^T$. 

**Reminder:**

If rotation around $v$ is $\omega \times$, then $v_3 = -\omega \times r$.

$$T_{01} = e^{[S_{01}]}$$
Forward kinematics: using screw motions

When only $\theta_3$ is allowed to move, we have

$$T_{04} = e^{[\mathcal{S}_3] \theta_3} M,$$

where, by definition,

$$[\mathcal{S}_3] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 & v_1 \\ \omega_3 & 0 & -\omega_1 & v_2 \\ -\omega_2 & \omega_1 & 0 & v_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus here,

$$[\mathcal{S}_3] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -(L_1 + L_2) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$
Forward kinematics: product of exponentials

We now repeat the procedure: assume $\theta_1$ fixed at zero, $\theta_3$ fixed at an arbitrary value, and move $\theta_2$. The corresponding twist is

$$\mathcal{S}_2 = (\omega_2 = (0, 0, 1)^T, (0, -L_1, 0))^T.$$

Thus $[\mathcal{S}_2] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -L_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. 

Again, $\omega = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and

$$\mathbf{v} = -\omega \times \mathbf{r} = \begin{bmatrix} 0 \\ -L_1 \\ 0 \end{bmatrix}.$$
Forward kinematics: product of exponentials

Now rotation about joint 2 can be viewed as applying a screw motion to the rigid body (link 1+ link 2), thus

\[ T_{14} = e^{[\mathcal{S}_2]_{\theta_2}} e^{[\mathcal{S}_3]_{\theta_3}} M. \]

Finally, \( \mathcal{S}_1 = ((0,0,1)^T, (0,0,0)^T) \) and

\[ T_{04} = e^{[\mathcal{S}_1]_{\theta_1}} e^{[\mathcal{S}_2]_{\theta_2}} e^{[\mathcal{S}_3]_{\theta_3}} M. \]

This is the product of exponentials representative of \( \Theta \rightarrow W \).
Product of exponentials: example

3R open chain, with non-collinear rotation axes.

Note: \( \hat{z}_i \) aligned with rotation axis, \( \hat{x}_i \) points to next joint.

Forward kinematics as the form

\[
T(\theta) = e^{[\mathcal{A}_1] \theta_1} e^{[\mathcal{A}_2] \theta_2} e^{[\mathcal{A}_3] \theta_3} M.
\]

We need to find \( M \) and the \( \mathcal{A}_i \)'s.
$M$ is the configuration of the end-effector fixed frame (frame 3) when all joint variables are zero. We obtain

$$M = \begin{bmatrix} 0 & 0 & 1 & L_1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$
The screw axis for joint 1 in frame 0 is $\mathcal{S}_1 = (\omega_1, v_1)$ with $\omega_1 = (0, 0, 1)^T$ and $v_1 = (0, 0, 0)^T$.

The screw axis for joint 2 in frame 0 is $\mathcal{S}_2 = (\omega_2, v_2)$ with $\omega_2 = (0, -1, 0)^T$ and $v_2 = (0, 0, -L_1)^T$.

To obtain $v_2$, set $q$ to be the vector joining origin of reference frame to center of joint, here $q = (L_1, 0, 0)^T$ and then $v_2 = \omega_2 \times (-q)$. Finally, $\mathcal{S}_3 = (\omega_3, v_3)$ with $\omega_3 = (1, 0, 0)^T$ and $v_3 = \omega_3 \times (- (L_1, 0, -L_2)^T) = (0, -L_2, 0)$.
Product of exponentials: example: RRPRRR open chain

Forward kinematics is described by

\[ M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & L_1 + L_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[
\begin{align*}
\mathbf{c}_1 &= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \\
\mathbf{c}_2 &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \\
\mathbf{c}_3 &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
\mathbf{c}_4 &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
\mathbf{c}_5 &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
\mathbf{c}_6 &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
\mathbf{t} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\end{align*}
\]

\[ T(\mathbf{c}, \mathbf{t}) = e^{\mathbf{c}} \cdots e^{\mathbf{c}_6} \]

\[
\begin{align*}
\mathbf{c}_1 &= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \\
\mathbf{c}_2 &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \\
\mathbf{c}_3 &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
\mathbf{c}_4 &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
\mathbf{c}_5 &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
\mathbf{c}_6 &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\end{align*}
\]

→ we expressed the screw vector of each link with respect to frame \(s\) and \(M\) is position of end effector in \(s\).
Recall the change of frame formula for twists: if $\mathcal{S}_1$ is the twist of link 1 in frame $s$ and $\mathcal{B}_1$ is the twist of link 1 in frame $b$, then

$$[\mathcal{S}_1] = T_{sb}[\mathcal{B}_1]T^{-1}_{sb} \quad \text{and} \quad [\mathcal{B}_1] = T_{bs}[\mathcal{S}_1]T^{-1}_{bs}.$$ 

or equivalently,

$$\mathcal{S}_1 = \text{Ad}_{T_{sb}}\mathcal{B}_1 \quad \text{and} \quad \mathcal{B}_1 = \text{Ad}_{T_{bs}}\mathcal{S}_1.$$ 

Recall that $M^{-1}e^AM = e^{M^{-1}AM}$. Thus

$$e^AM = Me^{M^{-1}AM}.$$ 

Algebraically, the construction relies on the conjugation rule:

$$e^AM = M^{-1}Ae^AM = Me^{M^{-1}AM} \quad \rightarrow \quad \text{commuting } e^Ae^M \text{ gives } M \text{ a conjugation}.$$