Introduction to Robotics
Lecture 1: Degrees of Freedom and Grübler formula
Configuration and DoFs

- Recall: we are dealing with the devices constructed by connecting **rigid bodies** (called **links**) using **joints**. A robot moves thanks to **actuators** providing forces and torques.

- Given that we know the shape of all links, joints and actuators, how can we describe every point in space that is part of the robot? This description is called the **configuration** of the robot (or C-space).

- Configuration of point in the plane is given by 2 coordinates: \((x, y)\). Configuration of a pinned rigid body on the plane is given by 1 coordinate. Configuration of an object in 3-spaces on a hinge has 1 coordinate.

Examples

C space exists on its own coordinates give names to points. C-spaces not always vector spaces; coordinates not always linear.
Every (almost) configuration space locally looks like a linear space.

The \textit{dimension} of the configuration space is the dimension of that approximation. (The number of coordinates one needs to describe unambiguously an arbitrary point of the space.)

It is also called the number of degrees of freedom (DoF) of the C-space.

A robot may have an end-effector, such as a hand or a gripper. The C-space of the end-effector is called the \textbf{task space} (or work space). It has a dimension (that is $\sum$ the dimension of the C-space).

One cannot have a global coordinate chart if C-space is not Euclidean; some singularities occur. Some latitude $\approx$ any latitude $\approx$ $90^\circ$ altitude. Same for W-space. So: prefer to work locally, employ charts once in a while.
How to fix position of a body in 2d: if you know distances between $A$, $B$ and $C$, and know where $A$ and $B$ go, you can find where $C$ is (almost).

Two points in plane, one constraint (distance between them): 3 dof.

Knowing the position of 3 *generic* points is sufficient to know the position of all points. More points will do as well.

General principle:

$$\text{DoF} = |\text{coordinates}| - |\text{independent constraints}|$$

We can use $(x_A, y_A, \theta)$ as *local* coordinates on the C-space.
Independent constraints

Introduce $D$: shall we consider distances to $A, B$ and $C$ to fix $D$?

No! the constraints are dependent (two fix the third one).

Given constraints

$$g_l(x_1, \ldots, x_d) = 0, \; 1 \leq l \leq m$$

where $g_l$ are differentiable functions, they are independent (at $x$) if the matrix of partial derivatives

$$\frac{\partial g}{\partial x} = \left( \frac{\partial g_i}{\partial x_j} \right)$$

is of full row rank.

Recall: A matrix $A \in \mathbb{R}^{n \times m}$ is of full row rank if its rows are linearly independent or, equivalently, if $\det(AB^\top) \neq 0$.

A rigid body in 3D has 6 degrees of freedom: 3 for translation and 3 for rotation. You can find this using the method above; we come back to this later.
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Independent constraints

How many DoF’s?

We need a systematic way of computing the number of DoFs.

Back to robots:
- links (rigid bodies)
- joints (constraining umbral behavior of these bodies)

Famous Δ-robot, invented to help chocolatiers.

Platform at bottom moves
A robot consists of rigid bodies attached with joints. Here are small zoo of joints:

Revolute (R)

Prismatic (P)

Helical (H)

Cylindrical (C)

Universal (U)

Spherical (S)

Given 2 planar rigid bodies in 3D, attach them with joint (X), how many degrees of freedom does the resulting robot have? Similarly, how many constraints between the rigid bodies does the joint impose?
Degrees of Freedom of a robot

<table>
<thead>
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<th>Joint type</th>
<th>dof</th>
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Orientation of the outgoing link in U joint is fixed.
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Grübler’s formula allows us to calculate in a systematic way the number of degrees of freedoms of a mechanism.

Let $J$ be the number of joints, $m$ be the number of degrees of freedom of a rigid body ($m = 3$ for planar mechanisms and $m = 6$ for spatial mechanisms).

**Proposition**
Consider a mechanism consisting of $N$ links, where ground is also regarded as a link. Let $f_i$ be the number of dofs provided by joint $i$, and $c_i$ be the number of constraints provided by joint $i$ (thus $f_i + c_i = m$ for all $i$). Then

$$
\# \text{dof} = m(N - 1) - \sum_{i=1}^{J} c_i \\
= m(N - 1) - \sum_{i=1}^{J} (m - f_i) \\
= m(N - 1 - J) + \sum_{i=1}^{J} f_i
$$

$N = 2 + 12 = 14$

$\# \text{dof} = 6(14 - 1 - 18) + 6 \cdot 1 + 12 \cdot 3 = -30 + 42 = 12 = 6 + 6$
Open- and closed-chain mechanisms

Closed-chain mechanism: mechanism with a closed-loop: e.g. both ends attached to the ground.

Open-chain or serial mechanism: mechanism without a closed-loop: e.g. links with a free end-effector.

\[ N = 6 \]
\[ J = 6 \]
\[ 6(6-1) + 6 \times 1 = 0 \]

SCARA, parallel: open chain

Triaps, or \( \Delta \) robot: closed loop

Classical area: linkages (going back to Watts regret)
Open- and closed-chain mechanisms