Introduction to Robotics
Lecture 0.5
Brief Introduction into Linear Algebra
Vectors

- Vector space - space whose elements one can add and multiply by a (real) number
- These elements are called vectors
- Special vector: the origin
- No other special vectors

Usual axioms:
- $p + q = q + p$
- $0 + p = p$
- $a(p + q) = ap + aq$

Many abstract linear (or vector) spaces

Examples of functions
Coordinates

Vector space can be equipped with \textit{linear coordinates}, i.e. functions such that

- They identify all points
- ...uniquely
- so that the coordinates of the sum of two points is the sum of coordinates (coordinates are \textit{linear} functions).

These functions are called a \textit{frame}.

In this case we can identify the points with the \textit{vector-columns}. 

\[
\begin{bmatrix}
\alpha \\
y
\end{bmatrix}(p) = \begin{bmatrix}
y(p+q) = \alpha(p) + \alpha(q)
\end{bmatrix}
\begin{bmatrix}
[a] \\
y
\end{bmatrix}(p) = \begin{bmatrix}
y_0 & y_1 & \cdots & y_n \\
\end{bmatrix}
\]

*non linear
coordinates
X(p) = \alpha(p)
-\alpha(p)
Y(p) = y(p)e
identify points, but
harder to work with

usual vector column
Important

Linear transformations appear in robotics in 2 ways:

- Going to different coordinates
- Moving a body

Changing the frame changes the coordinates of the point (not the point itself).

But those changes are linear.

...and linear transformations can encoded economically: once you know where the basis vectors go, you know everything.

\[
\begin{bmatrix}
\bar{x} \\
\bar{y}
\end{bmatrix}
(e_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\begin{bmatrix}
\bar{x} \\
\bar{y}
\end{bmatrix}
(e_2) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}
\begin{bmatrix}
\bar{x} \\
\bar{y}
\end{bmatrix}
(e_3) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}
\begin{bmatrix}
\bar{x} \\
\bar{y}
\end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
\]

Remember how to multiply matrices!

Different names for same point, related by linear transform:

\[
\begin{bmatrix}
x \\
y
\end{bmatrix}(P) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}(P) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}
X = x - y \\
Y = x + 2y \\
y = X - Y
\]

\( x = 2X + Y \)
Linear transformations are given by matrices, if you fixed the frame.

Composition of transformations leads to the product of matrices (eye-crossing operation).

Composition of linear transformations is linear again and is given by matrix \( BA \), note the order!

\[
\begin{bmatrix}
  f_1 \\
  \vdots \\
  f_k
\end{bmatrix} = \begin{bmatrix}
  f_1 \\
  \vdots \\
  f_k
\end{bmatrix}
\]

\[
A_p = A (x e_1 + y e_2) = x A e_1 + y A e_2 = a f_1 + y f_2
\]
Matrices: an example

In $\mathbb{R}^2$: 

$Ae_1 = A[e_1] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$Ae_2 = A[e_2] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$Ae_3 = A[e_3] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$A$ is given by matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

Sometimes matrices give you transformations that are not one-to-one.

$A: \mathbb{R}^d \to \mathbb{R}^d$ is 1-to-1 (each point has exactly 1 preimage) iff the image of $A$ is the whole space $\mathbb{R}^d$

The dimension of the image of $A$ is called its (column) rank
The criterion that a linear transformation is one-to-one is given by the *rank* of the matrix.

A square matrix has full rank iff its *determinant* is non-zero.

**Quiz**

Form $A = \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix}$, $B = \begin{vmatrix} d_5 & d_6 \\ d_7 & d_8 \end{vmatrix}$, find $\det A$, $\det B$, $\det(AB)$.
Euclidean motion is a transformation that preserves the lengths. Useful to describe rigid body motion! Length is a quadratic form, that is known as Pythagoras theorem.

Also gives you scalar product:

\[ u^\top v = \sum_k u_k v_k = |u||v| \cos \theta \]

\[ |u|^2 = u^\top u \]
Theorem

Moving a body rigidly (preserving the origin), changes its coordinates in a linear way.

So any Euclidean motion preserving the origin (aka a rotation) is linear and is given by a matrix $R$.

$$R = \begin{bmatrix} r_1 & r_2 & \cdots & r_d \end{bmatrix}$$

So any motion of a link is linear (if we fix a point on the link).
Euclidean Motions are Linear

Any rotation matrix is orthogonal, i.e.

\[ R^\top R = E, \]

where \( E \) is the identity matrix.
What if the Origin Moved?

Almost everything remains the same. But there are some dangers to beware:

- Shifting and rotating is not the same as rotating and shifting
- Classification of Euclidean motion becomes a bit more cumbersome

\[ T = s + Ap \]

In general:

\[ Tp = A(s+p) = As + Ap 
\]

Will come to it later