Linear Obstacles in Linear Systems, and Ways to Avoid Them

Yuliy Baryshnikov[*][†]

Department of Mathematics and Electrical and Computing Engineering,
University of Illinois, Urbana, IL, USA

January 3, 2022

1 Introduction

This note deals with the question of topology (meaning here, the homotopy type) of the space of trajectories of a control system avoiding certain obstacles. Such spaces often arise in the problems of motion planning or optimal control - that is the problems of finding a trajectory of an engineered system avoiding forbidden regions of the configuration space, and/or satisfying some optimality requirements, - and are common in robotics, control of autonomous systems, and many other areas, too numerous to list here.

1.1 Origins: Concurrency Problem

The original motivation for this research comes, however, from a special applied setting known in theoretical computer science under the moniker of *the spaces of directed paths*. There is a significant literature dealing with the topology of such spaces motivated by the *concurrency problem*.

Namely, consider joint execution of several algorithms (or processes) progressing asynchronously. The instantaneous state of the system is a vector $(x_1(t), \ldots, x_d(t))$ of states of each of the processes, which we can assume as taking values in the unit interval. The directed nature of the process implies that each x_l is non-decreasing in time.

Quite often these processes interact via some access restrictions: there might be some combinations of the states in some collections of processes which are not allowed (typically coming from the constraint that certain processes should not be allowed to claim the same resource at the same time: say, only one person at any given moment can withdraw from a particular account).

Geometrically, this restrictions translate into forbidden regions in the Cartesian products of execution spaces. In this picture the directed paths can be interpreted as the trajectories in $\mathbb{R}^d$ all of whose coordinates are increasing, or, if one assumes differentiability, as the trajectories whose velocities lie in the positive coordinate orthant, avoiding those forbidden regions.

The idea that the topology of the space of such obstacle-avoiding directed paths is highly relevant to the concurrency problem was realized early and addressed in abundant literature (see, e.g. [2]).

While the question about the structure of the space of directed paths in such continuous setting seems both natural and interesting, there are few explicit results about their homotopy type (see, however, [4, 3]).

1.2 Dynamical Systems

One can immediately generalize the problem from the setting where the trajectories have their speed vectors in the positive orthant to the trajectories whose derivatives are contained in a *convex open cone*. The structure

[*]Supported in part by ARO MURI SLICE and NSF DMS grant 1622370
[†]ymb.web.illinois.edu

of the cone is defined by the context: for example, the space-like trajectories in a Lorentzian space are (piece-wise) smooth trajectories $\mathbf{x}(t)$ satisfying $|\dot{\mathbf{x}}| \leq c$, or, equivalently, curves in the space-time $s \mapsto (\mathbf{x}, t)$ such that $dt/ds > 0$; $|d\mathbf{x}/ds| < c|dt/ds|$.

Another generalization is to look at the *control problem*: the linear span of admissible velocities need not necessarily be the full tangent space, but can be some subbundle of the tangent space.

In this note we will be dealing with the simplest class of such control systems, the *linear control systems* (introduced formally in Section 2.1).

Without the obstacles, there is no problem: the space of trajectories of a linear control system is convex. *With* the obstacles, the problem becomes more interesting, and was investigated by many authors in the case where the obstacles are stationary (or slowly moving). Here we will look into the variant where the obstacles are *time-dependent and fast or instantaneous*: how one would plan motion involving crossing a traffic light?

Of course, the problem can be addressed by extending the phase space to include the time coordinate. We would need, however, to restrict the trajectories to respect the direction of the time arrow: the time should increase along each trajectory.

1.3 Topology of the Spaces of Directed Paths

So, if there are some *obstacles*, i.e. subsets of the space-time that the trajectory needs to avoid, what is the homotopy type of the space of such directed paths?

We will address in this note the special case of linear control systems, with *instantaneous affine-linear obstacles*[1].

As usual, understanding this homotopy type is difficult in general, so that we focus on the characterization of common homotopy invariants, the (co)homology groups of the space.

We will be using throughout the singular (co)homologies with integer coefficients.

2 Setup

We start by discussing the spaces of trajectories we are interested in.

2.1 Linear Control Systems

Consider a linear control system,

$$\dot{\mathbf{x}} = A\mathbf{x} + B\boldsymbol{u}, \mathbf{x}(0) = \mathbf{x}_* \tag{1}$$

where $\mathbf{x} \in V = \mathbb{R}^d, \boldsymbol{u} \in P$, P a convex open subset of $U = \mathbb{R}^m$. Such systems are the staple of the state-based control theory (see e.g. [5]). The solutions of the system (1) are pairs of continuous, piece-wise differentiable *trajectories* $\mathbf{x} : [O, T] \to V$ and integrable *controls* $\boldsymbol{u} : [O, T] \to V$ that satisfy (1) almost everywhere. Equivalently, on can just consider P-valued controls on the interval where the system is defined, requesting it to be is regular enough (say, L_1 is sufficient).

The space $\mathbf{D}$ of trajectories $\mathbf{x}(\cdot)$ obtained this way is, obviously, convex, and its structure is of central importance to control theory.

We recall that the system (1) is *controllable* if the evaluation map sending a trajectory to its endpoint has an open range. For the linear time-invariant systems we consider here, controllability is equivalent to the fact that the A-hull of the image of B – i.e. the minimal A-invariant subspace of V containing the image of B, – is all of V.

We assume controllability throughout without the loss of generality: otherwise, we can just redefine V to be the A-hull of the image of B.

The key property of the controllable linear system is the following

[1] One can find many situations where this modeling assumption is relevant: think of an asteroid whose speed makes the duration of its presence in the flight path region negligible, but whose trajectory should be avoided at all costs…

Proposition 2.1. *Consider a trajectory* $\mathbf{x}$ *of* (1)*, an open subset* $A \subset [0,T]$ *of the time interval where the system is defined, and the space of trajectories* $\mathbf{D}_{U,\mathbf{x}}$ *matching* $\mathbf{x}$ *outside of* U*. If the system* (1) *is controllable, then for any finite subset* $\{t_1, \ldots, t_l\} \subset A$*, the evaluation map on* $\mathbf{D}_{U,\mathbf{x}}$ *given by*

$$e : \mathbf{x} \mapsto (\mathbf{x}(t_1), \ldots, \mathbf{x}(t_k))$$

is a submersion.

The result follows quite immediately from the standard computations, see e.g. [1].

2.2 Obstacles

Consider a sequence of *instantaneous obstacles* $\mathbf{O}_k \subset V, k = 1, \ldots, K$ realized at times $0 < t_1 < t_2 < \ldots < t_k < \ldots < t_K < T$. We will denote these data as $\mathcal{O} = \{(\mathbf{O}_k, t_k)\}_k$.

We will be interested in the trajectories of the system *avoiding the obstacles*:

$$\mathbf{D}_{\mathcal{O}} = \{(\mathbf{x}(\cdot) : \mathbf{x}(t_k) \notin \mathbf{O}_k \text{ for } k = 1, \ldots, K\}.$$

One can see easily that if the obstacles are closed subsets of V, then the collection of obstacle avoiding trajectories (identified with, say, the collection of the initial conditions and controls $\{\mathbf{x}_*, \boldsymbol{u}(\cdot)\} \subset V \times L_1([0,T], P)$ realizing obstacle avoiding paths) forms an open subset in the space of trajectories.

Throughout this note we will restrict ourselves to the simple situation of *affine-linear* obstacles $\mathbf{O}_k \subset V$.

We will be interested in the *homologies* of the space $\mathbf{D}_{\mathcal{O}}$ of obstacle-avoiding trajectories of (1).

3 Results

We start with a warm-up model, to illustrate some of the key notions.

3.1 Point Obstacles on Plane

Consider the simplest possible example (the *simple integrator*):

$$\dot{x} = u, x, u \in \mathbb{R}^1, x(0) = x_*, |u| < 1. \tag{2}$$

Consider a finite collection of 0-dimensional obstacles, - just a sequence of K points on the (t, x) plane ordered left-to-right:

$$\mathcal{O} = \{(t_1, x_1), (t_2, x_2), \ldots, (t_K, x_K)\}, 0 < t_1 < \ldots < t_K.$$

We can interpret the trajectories of this control system as Lipschitz ($L < 1$) functions, and the obstacle-avoiding trajectories are just the functions which don't pass through particular values at the specified times.

Proposition 3.1. *Obstacle avoiding trajectories for the system* (2) *form a finite collection of disjoint open convex subsets.*

Proof. Indeed, if two functions $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbf{D}_{\mathcal{O}}$ are in the same path-connected component, their values at each of the obstacle times t_l are in the same path-connected component of the real line from which the obstacles at that time are removed. As these components are convex, the linear homotopy between the functions $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ will avoid the obstacles. □

(Of course, this is true even if there are multiple point obstacles happening at the same instant.)

Therefore, the question on the topology of $\mathbf{D}_{\mathcal{O}}$ reduces to the question about the number of such components.

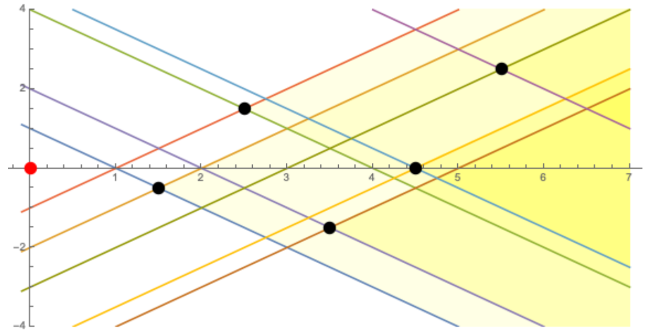

Figure 1: The simple linear control problem with point-like obstacles. The slope ± 1 functions through the obstacles are shown for visual help.

3.1.1 Example

Consider the collection of obstacles shown on Fig. 1 (here $x_* = 0$). One can easily see that there are 11 connected components of the obstacle-avoiding 1-Lipschitz functions satisfying $x(0) = 0$.

3.1.2 Chains

We augment the collection of obstacles with the point $(0, x_*)$, and define a *chain* to be a sequence of obstacles (starting with $\mathbf{O}_0 = (0, x_*)$) such that any two of them can be connected by a straight segment in the plane with the slope between -1 and 1, i.e. $|x_l - x_k| < t_l - t_k$ for $t_k < t_l$. Of course, these chains are the same as the chains in the partial order induced on the obstacles by $k \prec l \Leftrightarrow |x_l - x_k| < t_l - t_k$.

Theorem 3.2. *The space of obstacle avoiding trajectories in this system is a disjoint union of finitely many contractible components, which are in one-to-one correspondence with the* chains *of the ordering described above starting with* $(0, x_*)$.

Proof. Consider the decomposition of $\mathbf{D}_\mathcal{O} = \cup_\gamma \mathbf{D}_\mathcal{O}(\gamma)$ into open convex components. For each such component, the pointwise infimum of all trajectories in it (recall that we identify the trajectories with Lipschitz functions $\boldsymbol{x} : [0, T] \to \mathbb{R}$) is a piece-wise linear Lipschitz function $\boldsymbol{x}_\gamma(t) = \inf_{\boldsymbol{x} \in \mathbf{D}_\mathcal{O}(\gamma)} \boldsymbol{x}(t)$ passing through some collection of obstacles (including, necessarily, $\mathbf{O}_0 = (0, x_*)$). Those of the obstacles where $\boldsymbol{x}_\gamma$ is *not* locally linear form, necessarily, a chain, which we will refer to as the *marker* of the component. This gives a mapping from the components of $\mathbf{D}_\mathcal{O}$ to the chains. To reverse the correspondence, i.e. to associate to a chain a component, pick a small slack ϵ, and for a chain γ consider the function

$$\boldsymbol{x}_\gamma(t) := \max_{k \in \gamma} \left(x_k + \epsilon - (1 - \epsilon)|t - t_k| \right).$$

One can easily see that for small enough ϵ, this function will be in the component of $\mathbf{D}_\mathcal{O}$ whose marker coincides with the chosen chain. This bijection proves the statement. □

Returning to the example of Fig. 1: the partial order contains one 1-chain, $\mathbf{O}_0$, five 2-chains, $\mathbf{O}_0 \prec \mathbf{O}_k, k = 1, \ldots, 5$, five 3-chains $\mathbf{O}_0 \prec \mathbf{O}_1 \prec \mathbf{O}_3$, $\mathbf{O}_0 \prec \mathbf{O}_1 \prec \mathbf{O}_4$, $\mathbf{O}_0 \prec \mathbf{O}_1 \prec \mathbf{O}_5$, $\mathbf{O}_0 \prec \mathbf{O}_2 \prec \mathbf{O}_4$, $\mathbf{O}_0 \prec \mathbf{O}_2 \prec \mathbf{O}_5$ and no longer chains.

3.2 Main Result

Our main result deals with the situation where all obstacles are *affine linear* subspaces of V.

We need some preliminary definitions.

3.2.1 Avoidance classes

The space V from which an l-dimensional affine linear subspace $\mathbf{O}$ is removed is homotopy equivalent to the $c = d - l - 1$-dimensional sphere S^c. In particular, its c-dimensional cohomology group (reminder: all (co)-homology groups here are assumed over $\mathbb{Z}$) is cyclic, and generated by the class Poincare-dual to any properly cooriented $(l+1)$-dimensional affine half-plane in V having $\mathbf{O}$ as its boundary. We will be referring to this class as the *avoidance class* $a(\mathbf{O}) \in H^c(V - \mathbf{O})$.

We remark that for dimensional reasons, the self-products of the avoidance classes vanish, $a(\mathbf{O}) \smile a(\mathbf{O}) = 0$, unless the class is zero-dimensional, in which case $a(\mathbf{O}) \smile a(\mathbf{O}) = a(\mathbf{O})$.

Evaluating a trajectory in $\mathbf{D}_{\mathcal{O}}$ at the obstacle instances t_k generates the evaluation mapping

$$e_k : \mathbf{D}_{\mathcal{O}} \to V - \mathbf{O}_k.$$

Pull-backs of the avoidance classes to the space $\mathbf{D}_{\mathcal{O}}$ results in the cohomology classes

$$\alpha_k := e_k^*(a(\mathbf{O}_k)) \in H^{c_k}(\mathbf{D}_{\mathcal{O}})$$

(here $c_k = d - (\dim \mathbf{O}_k + 1)$ is the dimension of the avoidance class $a(\mathbf{O}_k)$). We will keep referring to those pullbacks as the avoidance classes.

3.2.2 Chains of obstacles

Generalizing the example of Section 3.1, we introduce *chains of obstacles*. Namely, an (ordered) sequence of obstacles $0 < \mathbf{O}_{k_1} < \mathbf{O}_{k_2} < \ldots < \mathbf{O}_{k_L}$ forms a *chain* if there exists a trajectory of (1) starting at at x_* and *passing* through all of the obstacles in the chain, for some admissible control $u(\cdot)$ in P.

Lemma 3.3. *The collections of trajectories passing through a chain of obstacles* $\mathbf{c}$ *is a convex subset of the space of trajectories.*

Proof. Indeed, it is the intersection of the convex space of solutions to (1) with the linear subspace of trajectories in passing through the obstacles at appropriate times. □

Viewed as a system of subsets of the collection of obstacles, the chains form, clearly, a simplicial complex (subset of a chain is a chain), which we will denote as $\mathbf{C}_{\mathcal{O}}$.

3.2.3 Example

Consider the *double integrator* system,

$$\dot{x} = v, \dot{v} = u; |u| < A \tag{3}$$

and a collection of the codimension 2 obstacles $\mathbf{O}_k = (x_k, v_k; t_k)$. Then the following holds:

Proposition 3.4. *The obstacles form a chain if for any pair of adjacent indices* $k < k'$ *in the sequence* $0 < k_1 < k_2 < \ldots$*, forms a chain, which leads to the natural compatibility condition:* $\mathbf{O}_k, \mathbf{O}_{k'}$ *is a chain* iff *the region in* (t, x) *plane given by*

$$\begin{aligned} t_k < t < t_{k'} \\ x_k + v_k(t - t_k) - A(t - t_k)^2/2 < x < x_k + v_k(t - t_k) + A(t - t_k)^2/2 \\ x_{k'} + v_{k'}(t - t_{k'}) - A(t - t_{k'})^2/2 < x < x_{k'} + v_{k'}(t - t_{k'}) + A(t - t_{k'})^2/2 \end{aligned} \tag{4}$$

is path-connected.

In particular, in this situation, the condition on the obstacles to form a chain is semi-algebraic.

Figure 2 presents an example of 3 obstacles (only (t, x) coordinates shown). The obstacles $\mathbf{O}_1$ at $t_1 = 0$, $\mathbf{O}_2$ at $t_2 = 3$, $\mathbf{O}_3$ at $t_3 = 4$ form two chains of length 2, $\mathbf{O}_1 \prec \mathbf{O}_2$ and $\mathbf{O}_1 \prec \mathbf{O}_3$. As the obstacles have dimension 0, a sequence of obstacles forms a chain iff any two neighboring obstacles form a chain. This is not true for the obstacles of higher dimensions.

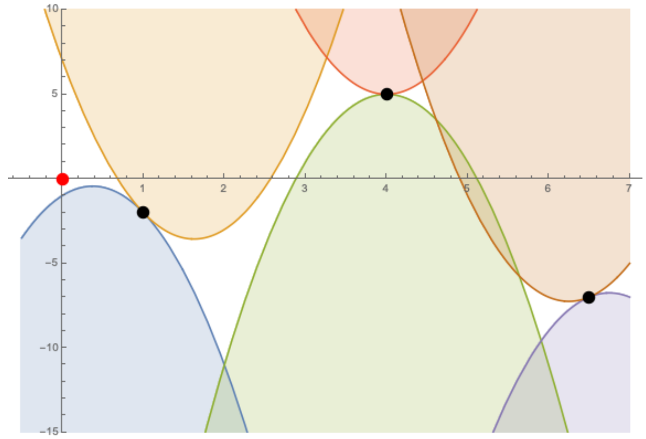

Figure 2: Two obstacles form a chain if the region bounded by the parabolas with given contact elements at the obstacles is connected.

Proof of Proposition 3.4. As the obstacles are of dimension 0, the condition that a time-ordered sequence of obstacles forms a chain depends only on any two consecutive obstacles $\mathbf{O}_k = (t_k, x_k, v_k)$ and $\mathbf{O}_{k'} = (t_{k'}, x_{k'}, v_{k'})$ in that sequence forming a chain. Clearly, the domain given by the condition (4) remains connected if A is decreased a bit. Take the shortest path within that domain connecting the points in the plane; the path consists of the segments of parabolas with the second derivative slightly less, in absolute value, than A, and straight segments, forming a continuously differentiable function, i.e. a trajectory solving (3) and connecting the obstacles $\mathbf{O}_k$ and $\mathbf{O}_{k'}$, which therefore form a chain. □

In general, the conditions on a sequence of obstacles to form a chain are not semialgebraic and can be quite complicated. Another special feature of the example above is the fact the obstacles are zero-dimensional: in this case, a collection of obstacles forms a chain if any two consecutive obstacles do, – that is not true in general.

3.2.4 Cohomology of the Space of Obstacle Avoiding Trajectories

Our main result describes completely the cohomology ring structure of the obstacle aviding trajectoris of (1): it is generated by the avoidance classes, and is subject by simple relations.

Definition 3.5. *Consider the graded ring generated by elements $\boldsymbol{o}_k$ of degree c_k (one generator for each obstacle; the degree equals the dimension of the corresponding avoidance class) and subject to the relations*

$$\begin{array}{l} \boldsymbol{o}_k^2 = 0 \textit{ unless } c_k = 0, \textit{ in which case } \boldsymbol{o}_k^2 = \boldsymbol{o}_k; \\ \boldsymbol{o}_{k_1} \cdots \boldsymbol{o}_{k_L} = 0 \textit{ unless the sequence of obstacles } \{\mathbf{O}_{k_l}\}, k_1 < \ldots < k_L \textit{ forms a chain.} \end{array} \tag{5}$$

This ring depends only on the combinatorial data: the simplicial complex of chains $\boldsymbol{\Delta}_{\mathcal{O}}$ and dimensions $c_k = d - (\dim \mathbf{O}_k + 1)$ and is denoted $\boldsymbol{R}_{\mathcal{O}}$.

On can immediately see that

Proposition 3.6. *The ring $\boldsymbol{R}_{\mathcal{O}}$ as an abelian group is freely generated by the products $\boldsymbol{o}_{k_1} \cdots \boldsymbol{o}_{k_L}$, where $\mathbf{O}_{k_1} \prec \cdots \prec \mathbf{O}_{k_L}$ is a chain.*

Our main result is the following

Theorem 3.7. *The cohomology ring $H^*(\mathbf{D}_{\mathcal{O}})$ is isomorphic to $\boldsymbol{R}_{\mathbf{O}}$ under the homomorphism sending each $\boldsymbol{o}_k$ to the corresponding avoidance class.*

3.2.5 Remarks

The ring $\boldsymbol{R}_{\mathcal{O}}$ is isomorphic to the Stanley-Reisner ring of the simplicial complex $\boldsymbol{\Delta}_{\mathcal{O}}$ factored by the squares of the generators of positive degree.

In the situation of the example of Section 3.1, all generators are in the dimension 0, so that the nontrivial elements of the cohomology ring span the space of functions constant on the components of $\mathbf{D}_{\mathcal{O}}$. This is essentially the 'algebra of the Heaviside functions' described in [6].

4 Proof of the Theorem 3.7

4.1 Finite-dimensional approximation

While the space $\mathbf{D}_{\mathcal{O}}$ is an infinite-dimensional (Banach) manifold, one can easily approximate it by a finite-dimensional manifold. We consider the (augmented) set of instances of obstacles as a mesh $\mathcal{T} := \{0 < t_1 < \ldots < t_K\} \subset [0, T]$ supporting the instances of the obstacles.

Consider the evaluation map taking the space $\mathcal{S}$ of solutions to (1) to $V^{\mathcal{T}}$: it sends a trajectory $\mathbf{x}(\cdot) \in \mathcal{S}$ to the collection of its values at times $t_k \in \mathcal{T}$. We will denote its image as $\mathcal{B} \subset V^{\mathcal{T}}$.

Proposition 4.1. *The evaluation map is a fibration with contractible fibers. In particular, its image $\mathcal{B}$ is a contractible open ball in $V^{\mathcal{T}}$, and its restriction to $\mathbf{D}_{\mathcal{O}}$ a (weak) homotopy equivalence to its image.*

Proof. Indeed, linear interpolations of the solutions to (1) (and their controls) are taken by the evaluation map to the linear interpolations between the points in $\mathcal{B}$. It implies the convexity of the image of the evaluation map; the fact that it is open follows from the controllability of the system (under which the evaluation map is a submersion). Further, the space of trajectories of (1) with fixed ends is an open convex subset of the space of trajectories, so that the evaluation map is a fibration with homotopy trivial fibers. □

4.2 Alexander duality

For each obstacle $\mathbf{O}_k$, we consider the (convex) set of solutions of (1) that *pass* through that obstacle. This is the intersection of the space of solutions to (1) with the preimage of $\mathbf{O}_k$ under the evaluation map e_k, a closed affine linear subspace of the space of trajectories. We will denote this subspace as $\mathbf{D}(\mathbf{O}_k)$, and its image in $\mathcal{B}$ as $\mathcal{B}(\mathbf{O}_k)$. Notice that that $\mathcal{B}(\mathbf{O}_k)$ is just the intersection of the open convex set $\mathcal{B}$ with the affine-linear subspace consisting of all points $p = (p_0, p_1, \ldots, p_K) \in V^{\mathcal{T}}$ with $p_k \in \mathbf{O}_k$ (compare with Lemma 3.3).

Now, by the Proposition 4.1, the space $\mathbf{D}_{\mathcal{O}}$ is homotopy equivalent to the complement to a subspace arrangement (i.e. a union of the affine-linear subspaces in Euclidean space) in the open convex disk $\mathcal{B}$:

$$\mathbf{D}_{\mathcal{O}} \cong \mathcal{B} - \mathcal{B}(\mathbf{O})$$

where $\mathcal{B}(\mathbf{O}) := \cup_k \mathcal{B}(\mathbf{O}_k)$.

We will identify the ambient space $\mathbb{R}^D \cong V^{\mathcal{T}} \supset \mathcal{B}$ of the finite-dimensional approximation of the space of trajectories with the punctured D-sphere. We denote by $K := S^D - \mathcal{B}$ the compact complement to the open disk $\mathcal{B}$, and by $K\mathcal{B}(\mathbf{O}) := K \cup \mathcal{B}(\mathbf{O})$ the union of that compact complement with the image of the obstacle $\mathbf{O}$ under the global evaluation map.

Then one has

Proposition 4.2. *There is an isomorphism of reduced homology groups*

$$\tilde{H}_*(\mathbf{D}_{\mathcal{O}}) \cong \tilde{H}^{D-*-1}(K \cup \mathcal{B}(\mathbf{O})).$$

Proof. Indeed,

$$\mathcal{B} - \mathcal{B}(\mathbf{O}) = S^D - (K \cup \mathcal{B}(\mathbf{O})),$$

whence the claim follows by Alexander duality. □

Further, we will notice that

$$\tilde{H}^{D-*-1}(K \cup \mathcal{B}(\mathbf{O})) \cong H^{D-*-1}(K \cup \mathcal{B}(\mathbf{O}), K), \tag{6}$$

as K is contractible.

4.3 Arrangements and Homotopy Limits

Now the problem falls squarely within the setup of the *homotopy limits of the arrangements of spaces* (see [8]).

Recall that a *diagram of spaces* consists of a finite partially ordered set (this level of generality more than suffices here) and a collection of assignments

1. of topological spaces $X(p)$ to each element p of the poset, and
2. of continuous mapping $d_{qp} : X(p) \to X(q)$ for each comparable pair $p \succ q$, satisfying the natural functorial properties, $d_{rq} \circ d_{qp} = d_{qr}$ for any chain $p \succ q \succ r$.

One can define in this situation a topological space $\|X(\cdot)\|$ (called *the homotopy limit* of the diagram of spaces $X(\cdot)$) and apply the methods of [8]. We will refer to that clearly written paper for the details of the constructions, as we will need just some of their implications.

One special case though, should be familiar: when all spaces $X(\cdot)$ are singletons, the homotopy limit is what is know as the *nerve* of the poset indexing the points, and consists of the simplices spanned by the chains in the poset (linearly ordered collections of elements).

Appropriating this setup to our situation we consider as the poset the collection $\mathbf{C}$ of chains of obstacles $\boldsymbol{c}$, ordered by reverse inclusion (i.e. $\boldsymbol{c} \succ \boldsymbol{c}'$ iff the latter is a subset of the former).

To construct the topological spaces associated to the chain we take the intersection

$$K\mathcal{B}(\mathbf{O}_{\boldsymbol{c}}) := \cap_{\mathbf{O}\in\boldsymbol{c}} K\mathcal{B}(\mathbf{O}),$$

and define $X(\boldsymbol{c})$ as the pair $(K\mathcal{B}(\mathbf{O}_{\boldsymbol{c}}), K)$. The mappings $d_{\boldsymbol{c}\boldsymbol{c}'} : X(\boldsymbol{c}') \to X(\boldsymbol{c})$ are just the inclusions. We will denote the resulting diagram of spaces as $K\mathbf{O}$.

Lemma 4.3. *The pair $(K\mathcal{B}(\mathbf{O}), K)$ is homotopy equivalent to $\|K\mathbf{O}\|$.*

Proof. From the construction, all the spaces $X(\boldsymbol{c})$ are unions of affine linear subspaces and a complement to an open convex subset in a Euclidean space. This immediately implies that any inclusion is a cofibration. Hence the reulst follows by the Projection lemma of [8]). □

The structure of the spaces $\|K\mathbf{O}\|$ is quite simple:

Lemma 4.4. *The homotopy limit $\|K\mathbf{O}\|$ is homotopy equivalent to the wedge*

$$(K \cup \mathcal{B}(\mathbf{O}_{\boldsymbol{c}}), K) \sim \left(\bigvee\nolimits_{\boldsymbol{c}\in\mathbf{C}} \|\mathbf{C}_{<\boldsymbol{c}}\| * S^{d(\boldsymbol{c})}, *\right) \tag{7}$$

where $\|\mathbf{C}_{<\boldsymbol{c}}\|$ is the nerve of the poset $\mathbf{C}_{<chain}$ of ordered by inclusion proper sub-chains of $\boldsymbol{c}$, and $d(\boldsymbol{c})$ is the dimension of $\mathcal{B}(\mathbf{O}_{\boldsymbol{c}})$.

Proof. Again, we basically repeat the argument of Theorem 2.2 in [8]): augment the poset $\mathbf{C}$ with the maximal element corresponding to K (the empty chain). All spaces $(K\mathcal{B}(\mathbf{O}_{\boldsymbol{c}'}), K)$ are homotopy equivalent to spheres mod a point, and the embeddings $d_{\boldsymbol{c}\boldsymbol{c}'}$ are homotopy equivalent to the constant maps. Hence, by the Wedge lemma (still, [8]), we obtain the claimed formula. □

We remark that $d(\boldsymbol{c})$ is given explicitely by

$$d(\boldsymbol{c}) = \sum_{s\in\mathcal{T}} d_s(\boldsymbol{c})$$

where $d_s(\boldsymbol{c}) = \dim(\mathbf{O}_k) =: d_k$ if $k \in \boldsymbol{c}$, and d otherwise.

It remains to describe $\|\mathbf{C}_{<\boldsymbol{c}}\|$, the nerve of the subposet of $\mathbf{C}$ spanned by the chains strictly contained in $\boldsymbol{c}$.

The instantaneous character of the obstacles implies the following

Proposition 4.5. *The spaces $\|\mathbf{C}_{<\boldsymbol{c}}\|$ are homeomorphic to $S^{|\boldsymbol{c}|-2}$: here $|\boldsymbol{c}|$ is the length of the chain $\boldsymbol{c}$ (by convention, $S^{-1} = \emptyset$).*

Proof. The arrangement of affine subspaces spanned by $\mathcal{B}(\mathbf{O}_k)$ in $V^{\mathcal{T}}$ is simple, i.e. the codimension of any nonempty intersection of these subspaces is the sum of the codimensions of these subspaces. This implies that the intersection poset $\mathbf{C}_{<\boldsymbol{c}}$ is isomorphic to the intersection poset of proper subsets of the chain $\boldsymbol{c}$, i.e. the poset of the boundary of $|\boldsymbol{c}|-1$-simplex. □

Collecting all elements together, we arrive at the

Proposition 4.6. *One has homotopy equivalence*

$$(K \cup \mathcal{B}(\mathbf{O}), K) \cong \big(\bigvee_{\boldsymbol{c}\in\mathbf{C}} S^{D-\sum_{k\in\boldsymbol{c}}(c_k-1)-1}, *\big)$$

(where $c_d = d - d_k$ is the codimension of $\mathbf{O}_k$).

4.3.1 Avoidance Cycles and Cocycles

To each chain of obstacles $\boldsymbol{c}$ one can associate a manifold $\gamma(\boldsymbol{c})$ in $\mathbf{D}_{\mathbf{O}}$ homeomorphic to the product of spheres

$$\gamma(\boldsymbol{c}) \cong \prod_{\mathbf{O}_k\in\boldsymbol{c}} S^{c_k-1},$$

where c_k is the codimension of $\mathbf{O}_k$ in V as follows.

Let $\mathbf{x} : [0, T] \to V$ is a trajectory of the linear systems, hitting the obstacles $\mathbf{O}_k$ which are part of the chain $\boldsymbol{c}$, but avoiding all other obstacles (if necessary, one can shift the trajectory $\mathbf{x}$ off the obstacles not in $\boldsymbol{c}$ by an arbitrarily small perturbation within $\mathbf{D}$, by Proposition 2.1).

Denote by U the union of disjoint open intervals around instances $\{t_k, k \in \boldsymbol{c}\}$ and apply Proposition 2.1: it would imply that for any k, there exists a small sphere S_k in $V - \mathbf{O}_k$ near $\mathbf{x}(t_k)$ evaluating to 1 on the avoidance class a_k and a family of trajectories deviating from $\mathbf{x}$ only in the vicinity of t_k, such that the composition of this family with the evaluation map e_k is the identity.

As these deformations agree outside of the intervals around instances $\{t_k, k \in \boldsymbol{c}\}$, one can embed the product of the spheres S_k to $\mathbf{D}_{\mathcal{O}}$; the resulting cycle (which we denote as $\gamma(\boldsymbol{c})$) will evaluate on the $\frown$-product of classes $\alpha_k, k \in \boldsymbol{c}$ to 1.

Finishing the proof of Theorem 3.7. We know already, that the dimension of the reduced cohomologies of $\mathbf{D}_{\mathcal{O}}$ are the same as the dimensions of the algebra $\boldsymbol{R}_{\mathcal{O}}$ (both possess bases enumerated by the chains of obstacles). Furthermore, there is a natural homomorphism from $\boldsymbol{R}_{\mathcal{O}}$ to $H^*(\mathbf{D}_{\mathcal{O}})$ sending the generator σ_k to the avoidance class α_k. The result would follow from the linear independence of the avoidance classes.

However, we constructed the cycles $\gamma(\boldsymbol{c})$ such that

$$(\alpha_{\boldsymbol{c}}, \gamma_{\boldsymbol{c}'}) = \delta_{\boldsymbol{c},\boldsymbol{c}'}. \tag{8}$$

Indeed, we know already that $(\alpha_{\boldsymbol{c}}, \gamma_{\boldsymbol{c}}) = 1$; if the dimensions of $\boldsymbol{c}, \boldsymbol{c}'$ are different, the pairing in (8) vanishes trivially. If these dimensiona are equal, and $\boldsymbol{c} \neq \boldsymbol{c}'$, then these two chains of obstacles differ at least at one obstacle (say, $\mathbf{O}_l \notin \boldsymbol{c}, \in \boldsymbol{c}'$). Then one can choose a halfspace representing a_l to be distinct from $\mathbf{x}(t_l)$, implying that the coupling vanishes as well. □

5 Examples and Conclusions

5.1 Examples

5.1.1 Codimension 1 obstacles

In this case the space of obstacle avoiding trajectories is a union of convex open components, and the only question is to enumerate those components. Our main result identifies the free abelian group generated by these components with the chains of obstacles, - the observation which can be observed by the elementary means in the warm-up example in the Section 3.1.

However, the result remains true in more complicated situations as well. Consider, for example, the double integrator example, and replace the obstacles with the condition of simply passing through the points (removing the slope condition). This will turn them into the codimension 1 obstacles. On one hand, this simplifies the topology of the components; on the other hand, the condition that a collection of obstacles forms a chain becomes more intricate, and involves interactions of obstacles beyond neighboring ones. Namely, a collection of obstacles forms a chain if there exists a quadratic spline (with the bound A on the absolute value of the second derivative) passing through the points of the chain. In the example shown on Fig. 2, such a spline exists for any collection of obstacles, and therefore the number of the connected components is 8.

5.1.2 Double Integrator, Codimension 2

In the example (3), the obstacles are of codimension 2 (to hit an obstacle the trajectory has to pass through the point, and match the slope), and so all the primitive avoidance classes $\mathbf{O}_k$ are of dimension 1. One can see that there are two nontrivial products of those classes (as evidenced by the fact there are 2 chains of length 3), so that the Poincare polynomial is

$$\sum_k \mathbf{rk} H^m(\mathbf{D}_\mathcal{O}) t^m = 1 + 3t + 2t^2.$$

In fact, one can see that $\mathbf{D}_\mathcal{O}$ is homotopy equivalent to the product of the wedge of two circles and S^1.

5.2 Concluding Remarks

The main theorem remains true (and its proof remains essentially the same) for *time-dependent* linear systems $\dot{\mathbf{x}} = A(t)\mathbf{x} + B(t)\boldsymbol{u}$, as long as the conclusion of the Proposition 2.1 remain true. For example, it is enough to request controllability over any open time interval containing more than one obstacle time.

Similarly, our theorem can be generalized to a significantly broader class of the obstacles. One simple generalization would include the 'portal'-like obstacles: complements in V to some open convex subsets.

Another, somewhat more demanding direction would deal with the obstacles which are themselves arrangements of subspaces [7]. Note that in this situation, if there is a single obstacle instance, and if the range of B in (1) is all of V, the space of the obstacle avoiding trajectories is retractable to the complement of the arrangement. So, our setup is a broad generalization of that classical problem. Still, applying the techniques similar to this paper would produce a reasonably complete answer (we will return to this class of problems elsewhere).

Far more interesting generalization deals with the obstacles that are non-instantaneous. The techniques of the diagrams of spaces would still apply, but the necessary modification would take much more room however, so we postpone them, as well as the detailed proofs of the results presented here to a later publication.

References

[1] A. A. Agrachev and Y. Sachkov, *Control theory from the geometric viewpoint*, vol. 87 of Encyclopaedia of Mathematical Sciences, Springer, 2013.

[2] L. Fajstrup, E. Goubault, E. Haucort, S. Mimram, and M. Raussen, *Directed algebraic topology and concurrency*, Springer, New York, NY, 2016.

[3] R. Meshulam and M. Raussen, *Homology of spaces of directed paths in Euclidean pattern spaces*, in A journey through discrete mathematics, Springer, 2017, pp. 593–614.

[4] M. Raussen and K. Ziemiański, *Homology of spaces of directed paths on Euclidean cubical complexes*, Journal of Homotopy and Related Structures, 9 (2014), pp. 67–84.

[5] E. D. Sontag, *Mathematical control theory*, vol. 6 of Texts in Applied Mathematics, Springer, second ed., 1998. Deterministic finite-dimensional systems.

[6] A. N. Varchenko and I. M. Gelfand, *Heaviside functions of a configuration of hyperplanes*, Funktsional. Anal. i Prilozhen., 21 (1987), pp. 1–18, 96.

[7] V. A. Vasiliev, *Topology of plane arrangements and their complements*, Uspekhi Mat. Nauk, 56 (2001), pp. 167–203.

[8] G. M. Ziegler and R. T. Živaljević, *Homotopy types of subspace arrangements via diagrams of spaces*, Mathematische Annalen, 295 (1993), pp. 527–548.